Chapter 2.4
Again iteration theory

As already mentioned, this book grew from an attempt to generalize the Wolff-Denjoy theorem to several variables, and this chapter is a report of the up-to-date situation in the field (at least as far as we know). We shall start with iteration theory on taut manifolds (of course); using the foundations laid down in chapters 2.1 and 2.3, it will be very easy to give a necessary and sufficient condition for the convergence of the sequence of iterates of a holomorphic map, already anticipated by Theorem 2.2.32. A little more work will be necessary to prove that if X is a taut manifold and $f \in \mathrm{Hol}(X, X)$ is such that $\{f^k\}$ is not compactly divergent, then $\{f^k\}$ is relatively compact in $\mathrm{Hol}(X, X)$; using this, we shall be able to characterize the set of limit points of $\{f^k\}$.

The complete generalization of the Wolff-Denjoy theorem, that is the description of the behavior of a compactly divergent sequence of iterates in a domain, is not so easy. Simple examples show that the naïve statement does not hold either in generic strongly pseudoconvex domains or in weakly convex domains, and it is then natural to wonder whether it is true at all, or whether it was a pure one-dimensional phenomenon. So, the most important result of this chapter is Theorem 2.4.23: if $D \subset\subset \mathbf{C}^n$ is a *strongly convex* domain and $f \in \mathrm{Hol}(D, D)$ has no fixed points, then the sequence of iterates of f converges, uniformly on compact sets, to a constant map $x \in \partial D$.

The proof of this theorem makes use of two distinct tools. The first one are the horospheres. We already saw in the first part of this book how to use Proposition 1.2.2 to define horocycles in multiply connected domains; now we shall do something very similar starting from Proposition 2.2.20, and replacing the Bergmann distance by the Kobayashi distance. Unfortunately, in general the limit (2.2.26) does not exist; so we are forced to introduce two families of horospheres, using $\liminf$ and $\limsup$ instead of the simple-minded limit. We shall be able to prove new versions of Julia's and Wolff's lemmas, and we shall use them respectively in the study of angular derivatives (in the last chapter) and iteration theory (guess where). Furthermore, the boundary behavior of the horospheres will be of the greatest importance: in fact, it will turn out that the lack of a Wolff-Denjoy theorem for weakly convex domains is due to the boundary shape of the horospheres.

The second tool is Theorem 2.4.20: if $D \subset\subset \mathbf{C}^n$ is a convex domain and f is a holomorphic map of D into itself, then $\{f^k\}$ is compactly divergent iff f has no fixed points. The proof of this theorem is a mixture of holomorphic (via the Kobayashi distance) and topological (via Brouwer's theorem) arguments, and indeed it seems that the lack of a Wolff-Denjoy theorem for generic (i.e., not homeomorphic to a ball) strongly pseudoconvex domains is due to topological reasons.

Using these tools we shall be able to study in detail iteration theory of holomorphic maps in convex domains. In particular, we shall prove the aforementioned generalization of the Wolff-Denjoy theorem in strongly convex domains, which is one of the highest peaks of this book.

2.4.1 *Taut manifolds*

We start this chapter by discussing, of course, iteration theory of holomorphic maps on taut manifolds. Let X be a taut manifold; we shall give a complete characterization of maps $f \in \mathrm{Hol}(X,X)$ such that the sequence of iterates $\{f^k\}$ converges in $\mathrm{Hol}(X,X)$; furthermore, we shall describe the set of limit points of $\{f^k\}$ in $\mathrm{Hol}(X,X)$ for a generic $f \in \mathrm{Hol}(X,X)$, and we shall also discuss the case of compact hyperbolic manifolds.

Clearly, the main tool here is Theorem 2.1.29, together with the related concepts of limit retraction and limit manifold. Using them, we can immediately give the characterization of maps with converging sequence of iterates:

Theorem 2.4.1: *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X,X)$. Then the sequence of iterates $\{f^k\}$ converges in $\mathrm{Hol}(X,X)$ iff f has a fixed point $z_0 \in X$ such that $\mathrm{sp}(df_{z_0}) \subset \Delta \cup \{1\}$.*

Proof: Assume first that the sequence $\{f^k\}$ converges, necessarily (by Theorem 2.1.29) to the limit retraction $\rho\colon X \to M$. Since

$$f \circ \rho = \lim_{k\to\infty} f \circ f^k = \lim_{k\to\infty} f^{k+1} = \rho,$$

f restricted to M is the identity. Take $z_0 \in M$; since $d\rho_{z_0} = \lim\limits_{k\to\infty} (df_{z_0})^k$, it follows that if λ is an eigenvalue of df_{z_0} then the sequence $\{\lambda^k\}$ of powers of λ converges to an element of $\mathrm{sp}(d\rho_{z_0}) \subset \{0,1\}$. Thus $\lambda \in \Delta \cup \{1\}$, and the first part of the assertion is proved.

Conversely, assume f has a fixed point $z_0 \in X$ such that $\mathrm{sp}(df_{z_0}) \subset \Delta \cup \{1\}$; in particular, $\{f^k\}$ is relatively compact in $\mathrm{Hol}(X,X)$. Let $\rho\colon X \to M$ be the limit retraction of f; note that M is taut, by Lemma 2.1.15. Arguing as in the proof of Theorem 2.2.32, let $T_{z_0}X = L_N \oplus L_U$ be the df_{z_0}-invariant splitting of $T_{z_0}X$ constructed in Theorem 2.1.21.(iv); note that, by Corollary 2.1.30, $L_U = T_{z_0}M$. Since, by hypothesis, $df_{z_0}|_{L_U} = \mathrm{id}$, it follows that $(df_{z_0})^k \to d\rho_{z_0}$ as $k \to +\infty$. In particular, z_0 is fixed by every limit point h of $\{f^k\}$, and $dh_{z_0} = d\rho_{z_0}$. Thus, by Theorem 2.1.21.(iii), $h|_M = \mathrm{id}_M$ and, by Theorem 2.1.29, $h = \rho$. In other words, ρ is the unique limit point of $\{f^k\}$ and, being $\{f^k\}$ relatively compact in $\mathrm{Hol}(X,X)$, $f^k \to \rho$, **q.e.d.**

So, again, maps with a fixed point arise on the scene. A natural question is: under what conditions does the sequence $\{f^k\}$ converge to a point $z_0 \in X$? The answer lies in the following definition: an *attractive fixed point* for a map $f \in \mathrm{Hol}(X,X)$ is a fixed point $z_0 \in X$ of f such that $\mathrm{sp}(df_{z_0}) \subset \Delta$. Then

Corollary 2.4.2: *Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$. Then $\{f^k\}$ converges to a point $z_0 \in X$ iff z_0 is an attractive fixed point for f.*

Proof: If z_0 is an attractive fixed point for f, then the limit multiplicity of f is 0 (by Corollary 2.1.30), and so $\{f^k\}$ converges to z_0. Conversely, if $\{f^k\}$ converges to $z_0 \in X$, then z_0 is the limit manifold of f, and the assertion again follows from Corollary 2.1.30, **q.e.d.**

Now we would like to describe the limit points of the sequence of iterates of a generic $f \in \mathrm{Hol}(X,X)$. If $\{f^k\}$ is compactly divergent, for the moment there is nothing to say. If $\{f^k\}$ is not compactly divergent, we know that every limit point in $\mathrm{Hol}(X,X)$ is of the form $\gamma \circ \rho$, where $\rho\colon X \to M$ is the limit retraction of f, and $\gamma \in \mathrm{Aut}(M)$. However, *a priori* there may be other limit points; *a priori*, the sequence $\{f^k\}$, which is not compactly divergent, could contain compactly divergent subsequences. In other words, it is conceivable that $\{f^k\}$ be not relatively compact in $\mathrm{Hol}(X,X)$. Fortunately, this is not the case, as shown in the following

Theorem 2.4.3: *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X,X)$ such that $\{f^k\}$ is not compactly divergent. Then $\{f^k\}$ is relatively compact in* $\mathrm{Hol}(X,X)$.

Proof: Let M be the limit manifold of f. To prove that $\{f^k\}$ is relatively compact in $\mathrm{Hol}(X,X)$, it clearly suffices to show that $\{(f|_M)^k\}$ has no compactly divergent subsequences; so, by Corollary 2.1.31, we can directly assume $X = M$, that is we can assume f is a pseudoperiodic automorphism of X.

Take $z_0 \in X$; it suffices to show that $A = \{f^k(z_0) \mid k \in \mathbf{N}\}$ is contained in a compact subset of X. Choose $\eta_0 > 0$ such that $\overline{B_k(z_0,\eta_0)}$ is compact, where we recall that $B_k(z_0,\eta_0)$ denotes the Kobayashi ball of center z_0 and radius η_0. Since $f \in \mathrm{Aut}(X)$, it follows that $B_k\big(f^h(z_0),\eta_0\big) \subset\subset X$ for every $h \in \mathbf{N}$. Now

$$\overline{B_k(z_0,\eta_0)} \subset B_k\big(B_k(z_0, 7\eta_0/8), \eta_0/4\big),$$

by Lemma 2.3.15; hence there are $w_1,\ldots,w_r \in B_k(z_0, 7\eta_0/8)$ such that

$$\overline{B_k(z_0,\eta_0)} \cap A \subset \bigcup_{j=1}^{r} B_k(w_j, \eta_0/4) \cap A,$$

and we can assume $B_k(w_j,\eta_0/4) \cap A \neq \emptyset$ for every $j = 1,\ldots,r$. For each $j = 1,\ldots,r$ take $k_j \in \mathbf{N}$ such that $f^{k_j}(z_0) \in B_k(w_j,\eta_0/4)$; then

$$B_k(z_0,\eta_0) \cap A \subset \bigcup_{j=1}^{r} \big[B_k\big(f^{k_j}(z_0), \eta_0/2\big) \cap A\big]. \tag{2.4.1}$$

Now, since f is pseudoperiodic, the set $\big\{k \in \mathbf{N} \bigm| k_X\big(f^k(z_0), z_0\big) < \eta_0/2\big\}$ is infinite; therefore we can find $k_0 \in \mathbf{N}$ such that

$$k_0 \geq \max\{1, k_1, \ldots, k_r\}, \tag{2.4.2}$$

$$k_X\big(f^{k_0}(z_0), z_0\big) < \eta_0/2. \tag{2.4.3}$$

Put

$$T = \bigcup_{h=1}^{k_0} \overline{B_k\big(f^h(z_0), \eta_0\big)};$$

since every $B_k\big(f^h(z_0),\eta_0\big)$ is relatively compact in X, it suffices to show that $A \subset T$.

Take $h \in \mathbf{N}$. If $h \le k_0$, then clearly $f^h(z_0) \in T$; so assume $h > k_0$. Choose $h_0 \ge h$ such that $k_X\big(f^{h_0}(z_0), z_0\big) < \eta_0/2$; hence, by (2.4.3), $k_X\big(f^{h_0}(z_0), f^{k_0}(z_0)\big) < \eta_0$ and

$$\forall\, 0 \le j \le k_0 \quad k_X\big(f^{h_0-j}(z_0), f^{k_0-j}(z_0)\big) = k_X\big(f^{h_0}(z_0), f^{k_0}(z_0)\big) < \eta_0.$$

In particular,

$$\forall h_0 - k_0 \le j \le h_0 \qquad\qquad f^j(z_0) \in T, \tag{2.4.4}$$

and $f^{h_0-k_0}(z_0) \in B_k(z_0,\eta_0) \cap A$. By (2.4.1) there is $1 \le l \le r$ such that

$$k_X\big(f^{k_l}(z_0), f^{h_0-k_0}(z_0)\big) < \eta_0/2,$$

and so

$$\forall\, 0 \le j \le \min\{k_l, h_0 - k_0\} \;\; k_X\big(f^{h_0-k_0-j}(z_0), f^{k_l-j}(z_0)\big) < \eta_0/2. \tag{2.4.5}$$

In particular, if $k_l \ge h_0 - k_0$ then, by (2.4.2), (2.4.4) and (2.4.5), it follows that $f^j(z_0) \in T$ for all $0 \le j \le h_0$.

If $k_l < h_0 - k_0$, set $h_1 = h_0 - k_0 - k_l$; then, by (2.4.2), $0 < h_1 < h_0$. Moreover, (2.4.2), (2.4.4) and (2.4.5) imply that $f^j(z_0) \in T$ for $h_1 \le j \le h_0$ and that $k_X\big(f^{h_1}(z_0), z_0\big) < \eta_0/2$. Then we can repeat the argument replacing h_0 by h_1, and in a finite number of steps we get $f^j(z_0) \in T$ for every $0 \le j \le h_0$. In particular, $f^h(z_0) \in T$; being h arbitrary, it follows that $A \subset T$, and the proof is complete, **q.e.d.**

Then let X be a taut manifold, take $f \in \mathrm{Hol}(X,X)$ such that the sequence $\{f^k\}$ is not compactly divergent, and denote by $\Gamma(f)$ the set of limit points of $\{f^k\}$ in $\mathrm{Hol}(X,X)$ (note that the closure of $\{f^k\}$ in $\mathrm{Hol}(X,X)$ is $\Gamma(f) \cup \{f^k\}$). By Theorem 2.4.3, $\Gamma(f)$ is a compact topological semigroup and, by Theorem 2.1.29, it is isomorphic to a compact topological semigroup of the automorphism group of the limit manifold of f. But even more is true:

Corollary 2.4.4: *Let X be a taut manifold, take $f \in \mathrm{Hol}(X,X)$ such that the sequence $\{f^k\}$ is not compactly divergent, and let $\rho\colon X \to M$ be its limit retraction. Then $\Gamma(f)$ is isomorphic to a compact abelian subgroup of $\mathrm{Aut}(M)$, which is the closed subgroup generated by $\varphi = f|_M \in \mathrm{Aut}(M)$.*

Proof: By Theorems 2.1.29, 2.4.3 and Corollary 2.1.31, it remains to show that for each limit point $h = \gamma \circ \rho \in \Gamma(f)$ we have $\gamma^{-1} \circ \rho \in \Gamma(f)$. But indeed fix a subsequence $\{f^{k_\nu}\}$ converging to ρ, and a subsequence $\{f^{m_\nu}\}$ converging to h. As usual, we can assume that $k_\nu - m_\nu \to +\infty$ and $f^{k_\nu - m_\nu} \to h_1 = \gamma_1 \circ \rho$ as $\nu \to +\infty$; then $h \circ h_1 = \rho = h_1 \circ h$, that is $\gamma_1 = \gamma^{-1}$, **q.e.d.**

In general, this is the best we can do. Indeed, let

$$D = \big\{(z, w) \in \mathbf{C}^2 \bigm| |z|^2 + |w|^2 + |w|^{-2} < 3\big\};$$

D is a strongly pseudoconvex domain, thus taut. Define $f \in \mathrm{Hol}(D, D)$ by

$$f(z, w) = (z/2, e^{i\theta} w^{\varepsilon}), \tag{2.4.6}$$

where $\varepsilon = \pm 1$ and $\theta \in \mathbf{R}$. Then the limit manifold of f is clearly the annulus

$$M = \big\{(0, w) \in \mathbf{C}^2 \bigm| |w|^2 + |w|^{-2} < 3\big\},$$

the limit retraction is $\rho(z, w) = (0, w)$ and choosing opportunely ε and θ we can obtain as $\Gamma(f)$ any compact abelian subgroup of $\mathrm{Aut}(M)$.

However, if f has a fixed point we can mime the proof of Proposition 2.2.33 to get some more information:

Corollary 2.4.5: *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X, X)$ with a fixed point $z_0 \in X$. Let $\lambda_1, \ldots, \lambda_r \in \partial\Delta$ be the eigenvalues of modulus 1 of df_{z_0}, listed according to their multiplicity; in particular, r is the limit multiplicity of f. Then $\Gamma(f)$ is isomorphic to a compact abelian group whose connected component at the identity is a real torus group of dimension at most r. To be precise, $\Gamma(f)$ is isomorphic to the closed subgroup of $\mathbf{T}^r$ generated by $(\lambda_1, \ldots, \lambda_r) \in \mathbf{T}^r$.*

Proof: By Theorem 2.1.29 and Corollary 2.1.22, every element of $\Gamma(f)$ is determined by the restriction to $T_{z_0}M$ of its differential at z_0, where M is the limit manifold of f. The assertion then follows from Theorem 2.1.21.(iv), Corollaries 2.1.30 and 2.4.4, and from the fact that every closed connected subgroup of $\mathbf{T}^r$ is a torus group of smaller dimension, **q.e.d.**

Note that this corollary allows us to describe $\Gamma(f)$ just looking at the spectrum of the differential of f at a fixed point; in particular, contrarily to Corollary 2.4.4, it is not necessary to know beforehand the limit manifold of f.

If X is compact, then no sequence in $\mathrm{Hol}(X, X)$ can be compactly divergent, of course. A consequence of this trivial observation is that on compact taut manifolds the iteration theory degenerates, as we shall now describe.

We need some information about the structure of $\mathrm{Aut}(X)$. The first fact is a very general theorem:

Theorem 2.4.6: *Let X be a compact complex manifold. Then $\mathrm{Aut}(X)$ is a complex Lie group.*

A proof is in Montgomery and Zippin [1955], for instance.

If X is a compact hyperbolic manifold, $\mathrm{Aut}(X)$ is a very particular kind of complex Lie group:

Corollary 2.4.7: *Let X be a compact hyperbolic manifold. Then* $\mathrm{Aut}(X)$ *is finite.*

Proof: First of all we prove that $\mathrm{Aut}(X)$ is discrete. If not, let $\gamma\colon \mathbf{C} \to \mathrm{Aut}(X)$ be a one-parameter subgroup; then for every $z_0 \in X$ the map $\zeta \mapsto \gamma(\zeta)z_0$ is a holomorphic map of $\mathbf{C}$ into X, and hence constant, for $k_{\mathbf{C}} \equiv 0$ and X is hyperbolic. Since this holds for all $z_0 \in X$, the only one-parameter subgroup of $\mathrm{Aut}(X)$ is the trivial one, and $\mathrm{Aut}(X)$ is discrete.

To end the proof it is enough to show that $\mathrm{Aut}(X)$ is compact. But X is complete hyperbolic, and thus taut; therefore (Proposition 2.1.24) $\mathrm{Aut}(X)$ is closed in $C^0(X, X)$. Since it is clearly equicontinuous with respect to k_X, by the Ascoli-Arzelà theorem $\mathrm{Aut}(X)$ is compact, **q.e.d.**

Another general fact we shall need is *Remmert's theorem:*

Theorem 2.4.8: *Let $f\colon X \to Y$ be a proper holomorphic map between two complex analytic spaces X and Y. Then $f(X)$ is a complex analytic subspace of Y.*

A proof can be found in Narasimhan [1966]; note that if X is compact then this theorem can be applied to any $f \in \mathrm{Hol}(X, Y)$, for in this case any element of $\mathrm{Hol}(X, Y)$ is automatically proper.

Using these powerful tools we can show why iteration theory on compact hyperbolic manifolds is void, generalizing Corollary 1.3.13:

Theorem 2.4.9: *Let X be a compact hyperbolic manifold, and $f \in \mathrm{Hol}(X, X)$. Then there exists $m \in \mathbf{N}^*$ such that f^m is a holomorphic retraction. In particular, the sequence of iterates of f converges iff f itself is a holomorphic retraction.*

Proof: Let $\rho\colon X \to M$ be the limit retraction of f. By Corollary 2.1.31, $f|_M \in \mathrm{Aut}(M)$; in particular, $f^k(X) \supset M$ for every $k \in \mathbf{N}$.

Now, $X \supset f(X) \supset f^2(X) \supset \cdots$ is a descending chain of compact hyperbolic analytic spaces; by Theorem 2.3.41 (cf. the notes to chapter 2.3), at every stage f is either a biholomorphism or everywhere degenerate. Therefore there is a $k_0 \in \mathbf{N}$ such that $f^{k+1}(X) = f^k(X) = M$ for all $k \geq k_0$.

By Corollary 2.4.7, $\mathrm{Aut}(M)$ is finite; so there is $m \geq k_0$ such that $(f|_M)^m = \mathrm{id}_M$. But then for any $z \in X$ we have

$$f^{2m}(z) = f^m\big(f^m(z)\big) = f^m(z),$$

since $f^m(z) \in M$, and we are done, **q.e.d.**

The next step in iteration theory is the study of the behavior of compactly divergent sequences of iterates in domains of $\mathbf{C}^n$. The candid hope in an immediate generalization of the Wolff-Denjoy theorem is undeceived at once: the map f defined in (2.4.6) has no fixed points and yet $\{f^k\}$ is not converging. An even worse example is the following: let $f \in \mathrm{Hol}(\Delta^2, \Delta^2)$ be given by

$$f(z, w) = \left(\frac{1+z}{3-z}, e^{i\theta} w\right),$$

for some $\theta \in \mathbf{R}$ with $e^{i\theta} \neq 1$. Then f is fixed point free, the sequence $\{f^k\}$ is even compactly divergent, and yet $\{f^k\}$ does not converge. As we shall see in the next two sections, this behavior is due to the existence of flat subsets of the boundary, and it will be excluded in strongly convex domains. But to explain all this, we need a new tool: the horospheres in general domains.

2.4.2 *Horospheres*

As already mentioned, the general definition of horosphere in an arbitrary bounded domain of $\mathbf{C}^n$ originated from the characterization of horospheres in B^n given in Proposition 2.2.20. In this section we shall introduce this new horospheres and prove their main properties; among them, new versions of Julia's and Wolff's lemma.

Let D be a bounded domain of $\mathbf{C}^n$, and choose $z_0 \in D$, $x \in \partial D$ and $R > 0$. Then the *small horosphere* $E_{z_0}(x,R)$ and the *big horosphere* $F_{z_0}(x,R)$ of *center* x, *pole* z_0 and *radius* R are defined by

$$\begin{aligned} E_{z_0}(x,R) &= \left\{ z \in D \;\middle|\; \limsup_{w\to x} \bigl[k_D(z,w) - k_D(z_0,w)\bigr] < \tfrac{1}{2}\log R \right\}, \\ F_{z_0}(x,R) &= \left\{ z \in D \;\middle|\; \liminf_{w\to x} \bigl[k_D(z,w) - k_D(z_0,w)\bigr] < \tfrac{1}{2}\log R \right\}. \end{aligned} \tag{2.4.7}$$

In (2.4.7), liminf and limsup are always finite. In fact, if z_0, z, $w \in D$, obviously

$$|k_D(z,w) - k_D(z_0,w)| \leq k_D(z_0,z);$$

hence for every $x \in \partial D$ we have

$$\begin{aligned} -\infty < -k_D(z_0,z) &\leq \liminf_{w\to x} \bigl[k_D(z,w) - k_D(z_0,w)\bigr] \\ &\leq \limsup_{w\to x} \bigl[k_D(z,w) - k_D(z_0,w)\bigr] \leq k_D(z_0,z) < +\infty. \end{aligned} \tag{2.4.8}$$

Some remarks about (2.4.7) are in order. First of all, the pole z_0 does not play any relevant role: we need it in the definition just as a normalization factor (cf. also Lemma 2.4.11). Secondly, our definition is not linked to $\mathbf{C}^n$; D could also be a hyperbolic manifold with boundary, and nothing would change. Thirdly, the natural setting for these definitions is within complete hyperbolic domains (cf. for instance Theorem 2.4.16 and the notes to this chapter); however, they make sense in general.

(2.4.8) implies several elementary properties of the horospheres, that we collect here for easy reference:

Lemma 2.4.10: *Let D be a bounded domain of $\mathbf{C}^n$, $z_0 \in D$ and $x \in \partial D$. Then:*

(i) *for every $R > 0$ we have $E_{z_0}(x,R) \subset F_{z_0}(x,R)$;*

(ii) *for every $0 < R_1 < R_2$ we have $E_{z_0}(x,R_1) \subset E_{z_0}(x,R_2)$ and $F_{z_0}(x,R_1) \subset F_{z_0}(x,R_2)$;*

(iii) *for every $R > 1$ we have $B_k(z_0, \frac{1}{2}\log R) \subset E_{z_0}(x,R)$;*

(iv) *for every $R < 1$ we have $F_{z_0}(x,R) \cap B_k(z_0, -\frac{1}{2}\log R) = \emptyset$;*

(v) $\bigcup_{R>0} E_{z_0}(x,R) = \bigcup_{R>0} F_{z_0}(x,R) = D$ *and* $\bigcap_{R>0} E_{z_0}(x,R) = \bigcap_{R>0} F_{z_0}(x,R) = \emptyset$.

Other useful properties are direct consequences of the definitions:

Lemma 2.4.11: *Let D be a bounded domain of $\mathbf{C}^n$, $z_0 \in D$ and $x \in \partial D$. Then:*
(i) *if $\varphi \in \mathrm{Aut}(D) \cap C^0(\overline{D})$, then for every $R > 0$*

$$\varphi\big(E_{z_0}(x,R)\big) = E_{\varphi(z_0)}\big(\varphi(x),R\big) \qquad \text{and} \qquad \varphi\big(F_{z_0}(x,R)\big) = F_{\varphi(z_0)}\big(\varphi(x),R\big);$$

(ii) *if $z_1 \in D$, set*

$$\tfrac{1}{2}\log L = \limsup_{w\to x}\big[k_D(z_1,w) - k_D(z_0,w)\big].$$

Then for every $R > 0$ we have $E_{z_1}(x,R) \subset E_{z_0}(x,LR)$ and $F_{z_1}(x,R) \subset F_{z_0}(x,LR)$.

Proof: (i) It is enough to remark that

$$k_D(z,w) - k_D(z_0,w) = k_D\big(\varphi(z),\varphi(w)\big) - k_D\big(\varphi(z_0),\varphi(w)\big),$$

and that $w \to x$ iff $\varphi(w) \to \varphi(x)$.
(ii) Setting

$$\begin{aligned} k_D(z,w) - k_D(z_0,w) &= \big[k_D(z,w) - k_D(z_1,w)\big] + \big[k_D(z_1,w) - k_D(z_0,w)\big],\\ k_D(z,w) - k_D(z_1,w) &= \big[k_D(z,w) - k_D(z_0,w)\big] + \big[k_D(z_0,w) - k_D(z_1,w)\big], \end{aligned}$$

and taking the $\limsup$ as $w \to x$ in the former equation and the $\liminf$ as $w \to x$ in the latter equation, we get

$$\begin{aligned} \limsup_{w\to x}\big[k_D(z,w) - k_D(z_0,w)\big] &\le \limsup_{w\to x}\big[k_D(z,w) - k_D(z_1,w)\big] + \tfrac{1}{2}\log L,\\ \liminf_{w\to x}\ \big[k_D(z,w) - k_D(z_1,w)\big] &\ge \liminf_{w\to x}\ \big[k_D(z,w) - k_D(z_0,w)\big] - \tfrac{1}{2}\log L, \end{aligned}$$

and the assertion follows, **q.e.d.**

In section 2.2.2, we studied the horospheres in the unit ball B^n. We saw that there the two families of horospheres coincide (i.e., the limit in (2.4.7) exists), and that a horosphere touches the boundary of B^n in just one point, namely the center. Now we want to study the horospheres in another model case, the unit polydisk Δ^n of $\mathbf{C}^n$, where the situation is very different.

Δ^n is the unit ball for the norm

$$\|z\| = \max\big\{|z_j| \bigm| j = 1,\dots,n\big\};$$

therefore (Corollary 2.3.7)

$$\forall z,w \in \Delta^n \qquad\qquad k_{\Delta^n}(z,w) = \tfrac{1}{2}\log\frac{1+\|\gamma_z(w)\|}{1-\|\gamma_z(w)\|},$$

where

$$\gamma_z(w) = \left(\frac{w_1 - z_1}{1-\overline{z_1}w_1},\dots,\frac{w_n - z_n}{1-\overline{z_n}w_n}\right)$$

is an automorphism of Δ^n with $\gamma_z(z) = 0$.

Since Δ^n is homogeneous, we may restrict ourselves to consider only horospheres with pole at the origin. Let $x \in \partial\Delta^n$; first of all

$$k_{\Delta^n}(z,w) - k_{\Delta^n}(0,w) = \log\left(\frac{1+\|\gamma_z(w)\|}{1+\|w\|}\right) + \frac{1}{2}\log\left(\frac{1-\|w\|^2}{1-\|\gamma_z(w)\|^2}\right).$$

Since $\|\gamma_z(x)\| = \|x\| = 1$, we just have to study the behavior of the second term. Now

$$1-\|w\|^2 = \min_h\{1-|w_h|^2\};$$
$$1-\|\gamma_z(w)\|^2 = \min_j\left\{\frac{1-|z_j|^2}{|1-\overline{z_j}w_j|^2}(1-|w_j|^2)\right\}.$$

Therefore

$$\frac{1-\|w\|^2}{1-\|\gamma_z(w)\|^2} = \min_h \max_j\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\cdot\frac{|1-\overline{z_j}w_j|^2}{1-|z_j|^2}\right\}. \tag{2.4.9}$$

Using (2.4.9) we may compute explicitely the horospheres. Since lim sup and max commute, we have

$$\begin{aligned}\limsup_{w\to x}\frac{1-\|w\|^2}{1-\|\gamma_z(w)\|^2} &= \max_j\left\{\limsup_{w\to x}\left[\frac{|1-\overline{z_j}w_j|^2}{1-|z_j|^2}\min_h\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\right\}\right]\right\}\\ &= \max_j\left\{\frac{|1-z_j\overline{x_j}|^2}{1-|z_j|^2}\limsup_{w\to x}\min_h\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\right\}\right\}.\end{aligned}$$

If $|x_j| < 1$, we have

$$\limsup_{w\to x}\frac{\min_h\{1-|w_h|^2\}}{1-|w_j|^2} = 0;$$

therefore we ought to consider only j's with $|x_j| = 1$. Furthermore,

$$\min_h\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\right\} \le 1;$$

hence

$$\limsup_{w\to x}\frac{1-\|w\|^2}{1-\|\gamma_z(w)\|^2} \le \max_j\left\{\frac{|1-z_j\overline{x_j}|^2}{1-|z_j|^2}\;\middle|\; |x_j| = 1\right\}. \tag{2.4.10}$$

We claim that (2.4.10) is an equality. To prove this, we need to exhibit a sequence $w^\nu \to x$ such that $(1-\|w^\nu\|^2)/(1-\|\gamma_z(w^\nu)\|^2)$ converges to the right-hand side of (2.4.10).

Set $w^\nu = (1-1/\nu)^{1/2}x$; we have

$$1-|(w^\nu)_h|^2 = (1-|x_h|^2) + |x_h|^2/\nu.$$

Hence if $|x_j| = 1$ we get

$$\lim_{\nu\to\infty} \min_h \left\{ \frac{1 - |(w^\nu)_h|^2}{1 - |(w^\nu)_j|^2} \right\} = 1,$$

and the claim follows.

To get the big horospheres, we proceed in the same way. Since lim inf and min commute, (2.4.9) yields

$$\liminf_{w\to x} \frac{1 - \|w\|^2}{1 - \|\gamma_z(w)\|^2} = \min_h \left\{ \liminf_{w\to x} \left[(1 - |w_h|^2) \max_j \left\{ \frac{|1 - z_j \overline{w_j}|^2}{(1 - |w_j|^2)(1 - |z_j|^2)} \right\} \right] \right\}.$$

If $|x_h| < 1$, we have

$$\liminf_{w\to x} \left[(1 - |w_h|^2) \max_j \left\{ \frac{|1 - z_j \overline{w_j}|^2}{(1 - |w_j|^2)(1 - |z_j|^2)} \right\} \right] = +\infty;$$

therefore we must again consider only h's with $|x_h| = 1$. Let us fix such a h; then

$$\liminf_{w\to x} \left[(1 - |w_h|^2) \max_j \left\{ \frac{|1 - z_j \overline{w_j}|^2}{(1 - |w_j|^2)(1 - |z_j|^2)} \right\} \right] \geq \liminf_{w\to x} \frac{|1 - z_h \overline{w_h}|^2}{1 - |z_h|^2} = \frac{|1 - z_h \overline{x_h}|^2}{1 - |z_h|^2};$$

hence

$$\liminf_{w\to x} \frac{1 - \|w\|^2}{1 - \|\gamma_z(w)\|^2} \geq \min_h \left\{ \frac{|1 - z_h \overline{x_h}|^2}{1 - |z_h|^2} \,\middle|\, |x_h| = 1 \right\}. \tag{2.4.11}$$

We claim that (2.4.11) is an equality too. This time we need to exhibit a sequence $w^\nu \to x$ such that $(1 - \|w^\nu\|^2)/(1 - \|\gamma_z(w^\nu)\|^2)$ converges to the right-hand side of (2.4.11). Choose h such that $|x_h| = 1$ and the quantity $|1 - z_h \overline{x_h}|^2/(1 - |z_h|^2)$ is minimal, and set

$$(w^\nu)_j = \begin{cases} (1 - 1/\nu)^{1/2} x_j, & \text{if } j \neq h; \\ (1 - 1/\nu^2)^{1/2} x_h, & \text{if } j = h. \end{cases}$$

If $j \neq h$, we get

$$\frac{1 - |(w^\nu)_h|^2}{1 - |(w^\nu)_j|^2} \cdot \frac{|1 - z_j \overline{(w^\nu)_j}|^2}{1 - |z_j|^2} = \frac{1/\nu^2}{(1 - |x_j|^2) + |x_j|^2/\nu} \cdot \frac{|1 - z_j \overline{(w^\nu)_j}|^2}{1 - |z_j|^2} \longrightarrow 0$$

as $\nu \to +\infty$; therefore

$$\lim_{\nu\to\infty} \max_j \left\{ \frac{1 - |(w^\nu)_h|^2}{1 - |(w^\nu)_j|^2} \cdot \frac{|1 - z_j \overline{(w^\nu)_j}|^2}{1 - |z_j|^2} \right\} = \frac{|1 - z_h \overline{x_h}|^2}{1 - |z_h|^2},$$

and the claim follows.

In conclusion, we have proved

Proposition 2.4.12: *Let $x \in \partial\Delta^n$ and $R > 0$. Then*

$$E_0(x,R) = \left\{ z \in \Delta^n \;\middle|\; \max_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\};$$

$$F_0(x,R) = \left\{ z \in \Delta^n \;\middle|\; \min_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\}.$$

To better understand this result, let us look at Δ^2. If we write $E^\Delta(\zeta, R)$ for the horocycle of center ζ and radius R in Δ, Proposition 2.4.12 says that typical examples of horospheres in Δ^2 are

$$E_0\big((1,1),R\big) = E^\Delta(1,R) \times E^\Delta(1,R), \qquad E_0\big((1,0),R\big) = E^\Delta(1,R) \times \Delta;$$
$$F_0\big((1,1),R\big) = \big(E^\Delta(1,R) \times \Delta\big) \cup \big(\Delta \times E^\Delta(1,R)\big), \qquad F_0\big((1,0),R\big) = E^\Delta(1,R) \times \Delta.$$

Figure 2.2 *The horospheres in Δ^2.*

Therefore in the polydisk small and big horospheres are really different, and they can touch the boundary in more than one point (see Figure 2.2). The exact situation is described by:

Corollary 2.4.13: *Let $x \in \partial\Delta^n$ and $R > 0$. Then:*
(i) *$E_0(x,R) = F_0(x,R)$ iff x has only one component of modulus 1;*
(ii) *$\overline{E_0(x,R)} \cap \partial\Delta^n = \{x\}$ iff x is on the Šilov boundary $(\partial\Delta)^n$ of Δ^n, while $\overline{F_0(x,R)} \cap \partial\Delta^n$ always contains $\{x\}$ properly.*

One may wonder if the properties of the horospheres described in Corollary 2.4.13 are due to the reducibility of Δ^n. This is not the case: the same situation presents itself in bounded symmetric domains. Indeed, let D be a bounded symmetric domain, and $x \in \partial D$. There exists a polydisk $\Delta^r \subset D$ (where r is the rank of the domain) such that $x \in \partial\Delta^r \subset \partial D$ and $k_{\Delta^r} = k_D|_{\Delta^r \times \Delta^r}$ (see Wolf [1972] and Abate [1987]). Hence

$$E_0^D(x,R) \cap \Delta^r \subset E_0^\Delta(x,R) \qquad \text{and} \qquad F_0^D(x,R) \cap \Delta^r \supset F_0^\Delta(x,R),$$

where E^Δ, E^D (F^Δ, F^D) are the small (big) horospheres of Δ^r, respectively D. Therefore, by Corollary 2.4.13.(i), $E_0^D(x,R)$ and $F_0^D(x,R)$ are, in general, different and, by Corollary 2.4.13.(ii), $F_0^D(x,R)$ always touches the boundary in a set strictly bigger than $\{x\}$.

The latter phenomenon is somehow originated by the flatness of the boundary of the polydisk. Our next aim is to prove that in strongly pseudoconvex domains this does not occur:

Theorem 2.4.14: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain. Then for every $z_0 \in D$, $x \in \partial D$ and $R > 0$*

$$\overline{E_{z_0}(x,R)} \cap \partial D = \overline{F_{z_0}(x,R)} \cap \partial D = \{x\}.$$

Proof: We begin proving that x belongs to the closure of $E_{z_0}(x,R)$. Let $\varepsilon > 0$ be given by Theorem 2.3.56; then, recalling Theorem 2.3.52, for every z, $w \in B(x,\varepsilon)$ we have

$$k_D(z,w) - k_D(z_0,w) \le \tfrac{1}{2}\log\left(1 + \frac{\|z-w\|}{d(z,\partial D)}\right) + \tfrac{1}{2}\log\bigl[d(w,\partial D) + \|z-w\|\bigr] + K,$$

for a suitable constant $K \in \mathbf{R}$ depending only on x and z_0. In particular, if $\|z-x\| < \varepsilon$ we get

$$\limsup_{w\to x}[k_D(z,w) - k_D(z_0,w)] \le \tfrac{1}{2}\log\left(1 + \frac{\|z-x\|}{d(z,\partial D)}\right) + \tfrac{1}{2}\log\|z-x\| + K. \qquad (2.4.12)$$

Hence if we take a sequence $\{z_\nu\} \subset D$ converging to x so that $\{\|z_\nu - x\|/d(z_\nu,\partial D)\}$ is bounded (for instance, a sequence converging non-tangentially to x), then for every $R > 0$ we have $z_\nu \in E_{z_0}(x,R)$ eventually, and thus $x \in \overline{E_{z_0}(x,R)}$.

To conclude the proof, we have to show that x is the only boundary point belonging to the closure of $F_{z_0}(x,R)$. Suppose, by contradiction, that there exists $y \in \partial D \cap \overline{F_{z_0}(x,R)}$ with $y \ne x$; then we can find a sequence $\{z_\mu\} \subset F_{z_0}(x,R)$ with $z_\mu \to y$.

Corollary 2.3.55 provides us with $\varepsilon > 0$ and $K \in \mathbf{R}$ associated to the pair (x,y); we may assume $\|z_\mu - y\| < \varepsilon$ for all $\mu \in \mathbf{N}$. Since $z_\mu \in F_{z_0}(x,R)$, we have

$$\forall \mu \in \mathbf{N} \qquad \liminf_{w\to x}\bigl[k_D(z_\mu,w) - k_D(z_0,w)\bigr] < \tfrac{1}{2}\log R;$$

therefore for each $\mu \in \mathbf{N}$ we can find a sequence $\{w_{\mu\nu}\} \subset D$ such that $\lim\limits_{\nu\to\infty} w_{\mu\nu} = x$ and

$$\lim_{\nu\to\infty}\bigl[k_D(z_\mu,w_{\mu\nu}) - k_D(z_0,w_{\mu\nu})\bigr] < \tfrac{1}{2}\log R.$$

Moreover, we can assume $\|w_{\mu\nu} - x\| < \varepsilon$ and $k_D(z_\mu,w_{\mu\nu}) - k_D(z_0,w_{\mu\nu}) < \frac{1}{2}\log R$ for all μ, $\nu \in \mathbf{N}$.

By Corollary 2.3.55, for all μ, $\nu \in \mathbf{N}$ we have

$$\begin{aligned}\tfrac{1}{2}\log R &> k_D(z_\mu,w_{\mu\nu}) - k_D(z_0,w_{\mu\nu})\\ &\ge -\tfrac{1}{2}\log d(z_\mu,\partial D) - \tfrac{1}{2}\log d(w_{\mu\nu},\partial D) - k_D(z_0,w_{\mu\nu}) - K.\end{aligned}$$

On the other hand, the upper boundary estimate (Theorem 2.3.51) yields $c_1 > 0$ (independent of $w_{\mu\nu}$) such that

$$\forall \mu, \nu \in \mathbf{N} \qquad k_D(z_0, w_{\mu\nu}) \le c_1 - \tfrac{1}{2} \log d(w_{\mu\nu}, \partial D).$$

Therefore

$$\forall \mu \in \mathbf{N} \qquad \tfrac{1}{2} \log R > -\tfrac{1}{2} \log d(z_\mu, \partial D) - K - c_1,$$

and, letting μ go to infinity, we get a contradiction, **q.e.d.**

As we shall see in the next section, this theorem is one of the main reasons behind the different behavior of iterates in strongly convex domains with respect to weakly convex domains.

In the first part of this book, we saw that in the classical theory of horocycles a preminent position is occupied by Julia's lemma. For our horospheres, this is absolutely natural: exactly as the Kobayashi distance has a built-in Schwarz lemma, so our horospheres have a built-in Julia lemma.

To introduce this new version, look at Theorem 2.2.21, Julia's lemma in B^n. That theorem can be applied to maps $f \in \mathrm{Hol}(B^n, B^n)$ such that $d\big(f(z), \partial B^n\big)/d(z, \partial B^n)$ has finite $\liminf$ as $z \to x \in \partial B^n$. Now, in section 2.3.5 we have learned that in a strongly pseudoconvex domain D with base point z_0 the quantity $-\frac{1}{2} \log d(z, \partial D)$ is essentially the same as $k_D(z_0, z)$; so the hypothesis in a Julia lemma can be something like

$$\liminf_{w \to x} \big[k_D(z_0, w) - k_D\big(z_0, f(w)\big)\big] < +\infty. \tag{2.4.13}$$

Note that since

$$k_D(z_0, w) - k_D\big(z_0, f(w)\big) \ge k_D\big(f(z_0), f(w)\big) - k_D\big(z_0, f(w)\big) \ge -k_D\big(z_0, f(z_0)\big),$$

the $\liminf$ in (2.4.13) is never $-\infty$. Moreover, if that $\liminf$ is finite for one point $z_0 \in D$, then it remains finite replacing z_0 by any other point of D.

Then our most general version of Julia's lemma is:

Proposition 2.4.15: *Let D be a bounded domain of $\mathbf{C}^n$, and fix a point $z_0 \in D$. Let $f \colon D \to D$ be a holomorphic map such that there is a sequence $\{w_\nu\} \subset D$ converging to $x \in \partial D$ so that $\{f(w_\nu)\}$ converges to a point $y \in \partial D$ and*

$$\lim_{\nu \to \infty} \big[k_D(z_0, w_\nu) - k_D\big(z_0, f(w_\nu)\big)\big] \le \tfrac{1}{2} \log \alpha < +\infty,$$

for a suitable $\alpha \in \mathbf{R}^+$. Then

$$\forall R > 0 \qquad f\big(E_{z_0}(x, R)\big) \subset F_{z_0}(y, \alpha R).$$

Proof: Let $z \in E_{z_0}(x,R)$; then

$$\begin{aligned}
&\liminf_{w\to y}\bigl[k_D(f(z),w) - k_D(z_0,w)\bigr] \\
&\qquad\le \liminf_{\nu\to\infty}\bigl[k_D\bigl(f(z),f(w_\nu)\bigr) - k_D\bigl(z_0,f(w_\nu)\bigr)\bigr] \\
&\qquad\le \liminf_{\nu\to\infty}\bigl[k_D(z,w_\nu) - k_D\bigl(z_0,f(w_\nu)\bigr)\bigr] \\
&\qquad\le \liminf_{\nu\to\infty}\bigl[k_D(z,w_\nu) - k_D(z_0,w_\nu)\bigr] + \lim_{\nu\to\infty}\bigl[k_D(z_0,w_\nu) - k_D\bigl(z_0,f(w_\nu)\bigr)\bigr] \\
&\qquad\le \limsup_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + \tfrac{1}{2}\log\alpha \\
&\qquad< \tfrac{1}{2}\log(\alpha R),
\end{aligned}$$

q.e.d.

However, as already anticipated, the horospheres are really meaningful only in complete hyperbolic domains (if $x \in \partial D$ is at finite Kobayashi distance from D, then the horospheres centered at x are just Kobayashi balls of center x). Accordingly, in complete hyperbolic domains we have a neater statement:

Theorem 2.4.16: *Let $D \subset\subset \mathbf{C}^n$ be complete hyperbolic. Let $f \in \mathrm{Hol}(D,D)$ and $x \in \partial D$ be such that there are $z_0 \in D$ and $\alpha \in \mathbf{R}^+$ so that*

$$\liminf_{w\to x}\bigl[k_D(z_0,w) - k_D\bigl(z_0,f(w)\bigr)\bigr] \le \tfrac{1}{2}\log\alpha.$$

Then there exists $y \in \partial D$ such that

$$\forall R > 0 \qquad f\bigl(E_{z_0}(x,R)\bigr) \subset F_{z_0}(y,\alpha R). \tag{2.4.14}$$

Furthermore, if D is strongly pseudoconvex then y is unique and f has non-tangential limit y at x.

Proof: Choose a sequence $\{w_\nu\} \subset D$ converging to x such that

$$\lim_{\nu\to\infty}\bigl[k_D(z_0,w_\nu) - k_D\bigl(z_0,f(w_\nu)\bigr)\bigr] = \liminf_{w\to x}\bigl[k_D(z_0,w) - k_D\bigl(z_0,f(w)\bigr)\bigr];$$

up to a subsequence, we can also assume $f(w_\nu) \to y \in \overline{D}$. Since D is complete hyperbolic, $k_D(z_0,w_\nu) \to +\infty$ as $\nu \to +\infty$; therefore $k_D\bigl(z_0,f(w_\nu)\bigr) \to +\infty$ as well, y must belong to ∂D and we can apply Proposition 2.4.15.

Finally, assume D strongly pseudoconvex, and choose $y \in \partial D$ such that (2.4.14) holds; it suffices to show that y is the non-tangential limit of f at x.

Let $\{z_\nu\} \subset D$ be a sequence converging non-tangentially to x. Then (2.4.12) implies that for every $R > 0$ we have $z_\nu \in E_{z_0}(x,R)$ eventually, and thus $f(z_\nu) \in F_{z_0}(y,\alpha R)$ eventually. In particular, Lemma 2.4.10.(iv) yields $f(z_\nu) \notin B_k(z_0, -\frac{1}{2}\log R)$ eventually, and so every limit point of $\{f(z_\nu)\}$ belongs to the boundary of D (for D is complete hyperbolic or, if you prefer, because of Theorem 2.3.52). But then we can quote Theorem 2.4.14, stating that the unique point belonging both to ∂D and to the closure of $F_{z_0}(y,\alpha R)$ is y itself, concluding that $f(z_\nu) \to y$ and, by the arbitrariness of $\{z_\nu\}$, the assertion, **q.e.d.**

If D is not strongly pseudoconvex, the point y may not be uniquely determined, essentially because two distinct points on ∂D can have the same horospheres; cf. Proposition 2.4.12.

As customary, Julia's lemma will be used to study angular derivatives, as we shall see in chapter 2.7, after the introduction of new technical tools to be defined in chapter 2.6; it is now time to talk about Wolff's lemma.

The standard Wolff lemma in B^n (and Δ) was applied to a map without fixed points. Our first Wolff's lemma requires a bit more: we need that the sequence of iterates of the map under consideration be compactly divergent. In B^n this is equivalent to having no fixed points (Proposition 2.2.30), but in general this is not enough, as shown by the map (2.4.6). Anyway, we can prove the following

Proposition 2.4.17: *Let $D \subset\subset \mathbf{C}^n$ be a complete hyperbolic domain with simple boundary, $z_0 \in D$ and $f \in \mathrm{Hol}(D, D)$ such that $\{f^k\}$ is compactly divergent. Then there exists $x \in \partial D$ such that*

$$\forall R > 0 \qquad f\big(E_{z_0}(x, R)\big) \subset F_{z_0}(x, R).$$

In particular, if D is strongly pseudoconvex then f has non-tangential limit x at x.

Proof: Since $\{f^k\}$ is compactly divergent and D is complete hyperbolic,

$$\lim_{k\to\infty} k_D\big(z_0, f^k(z_0)\big) = +\infty.$$

Now, we claim there is a subsequence $\{f^{k_\nu}\}$ such that

$$\forall \nu \in \mathbf{N} \qquad k_D\big(z_0, f^{k_\nu}(z_0)\big) < k_D\big(z_0, f^{k_\nu+1}(z_0)\big). \tag{2.4.15}$$

Indeed, let k_ν denote the largest integer k satisfying $k_D\big(z_0, f^k(z_0)\big) \le \nu$; then

$$k_D\big(z_0, f^{k_\nu}(z_0)\big) \le \nu < k_D\big(z_0, f^{k_\nu+1}(z_0)\big).$$

Up to a subsequence, we can assume that $f^{k_\nu} \to x \in \partial D$ (for $\{f^{k_\nu}\}$ is compactly divergent and D is bounded with simple boundary). In particular, if we set $w_\nu = f^{k_\nu}(z_0)$, we have $w_\nu \to x$, $f(w_\nu) = f^{k_\nu}\big(f(z_0)\big) \to x$ and

$$\liminf_{\nu\to\infty} \big[k_D(z_0, w_\nu) - k_D\big(z_0, f(w_\nu)\big)\big] \le 0,$$

by (2.4.15). Then we can apply Proposition 2.4.15 with $\alpha = 1$ and $y = x$, and the first assertion follows. The last assertion is an immediate consequence of Theorem 2.4.16, **q.e.d.**

This proposition has one main disadvantage: since, in general, $F_{z_0}(x, R)$ is strictly bigger than $E_{z_0}(x, R)$, Proposition 2.4.17 gives no information about $f^k\big(E_{z_0}(x, R)\big)$ for $k \ge 2$, and so it is not the right statement for iteration theory.

We can obviate this disadvantage (even reintroducing the hypothesis of no fixed points) replacing the condition of simple boundary by a (strong) internal condition that we shall now describe. Let $D \subset\subset \mathbf{C}^n$ be a bounded domain; we shall say that D is *compactly approximable* if there exists a sequence $\{g_\nu\}$ of holomorphic maps of D into itself converging to the identity of D such that $g_\nu(D) \subset\subset D$ for every $\nu \in \mathbf{N}$. For instance, every convex domain D is compactly approximable: it suffices to choose $z_0 \in D$, a sequence $\{r_\nu\} \subset (0,1)$ converging to 1 and set $g_\nu(z) = z_0 + r_\nu(z - z_0)$. More generally, every bounded domain D star-shaped with respect to a point $z_0 \in D$ and such that $z_0 + r(x - z_0) \in D$ for all $x \in \partial D$ and $r \in (0,1)$ close enough to 1 is compactly approximable.

A compactly approximable domain is topologically fairly trivial:

Lemma 2.4.18: *Let $D \subset\subset \mathbf{C}^n$ be a compactly approximable domain. Then all the homology groups $H_k(D, \mathbf{Z})$ and the homotopy groups $\pi_k(D)$ vanish.*

Proof: Let $\gamma \in H_k(D, \mathbf{Z})$; we can find a representative of γ compactly supported in D. Therefore there is ν large enough such that $(g_\nu)_*\gamma = \gamma$. Now, by Corollary 2.1.32, the sequence $\{(g_\nu)^k\}$ converges to a point $z_0 \in D$. But then, for k large enough, $\gamma = (g_\nu)_*^k\gamma$ is contained in a contractible neighbourhood of z_0, and so $\gamma = 0$. The same argument applies to the homotopy groups, **q.e.d.**

It is then natural to conjecture that every strongly pseudoconvex topologically contractible domain is compactly approximable; unexpectedly, this is not true. In fact, Lin and Zaĭdenberg [1979] have constructed a bounded strongly pseudoconvex domain with analytic boundary homeomorphic to the ball not compactly approximable.

Anyway, convex domains provide a sufficiently large supply of compactly approximable domains to justify the importance of the following version of Wolff's lemma:

Theorem 2.4.19: *Let $D \subset\subset \mathbf{C}^n$ be a compactly approximable domain, and take a map $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x \in \partial D$ such that*

$$f^k\bigl(E_{z_0}(x, R)\bigr) \subset F_{z_0}(x, R)$$

for every $z_0 \in D$, $R > 0$ and $k \in \mathbf{N}$.

Proof: Let $\{g_\nu\}$ be a sequence of maps converging to id_D such that $g_\nu(D) \subset\subset D$ for every $\nu \in \mathbf{N}$. Set $f_\nu = g_\nu \circ f$. By Corollary 2.1.32, every f_ν has a fixed point $w_\nu \in D$. Up to a subsequence, we may assume that $\{w_\nu\}$ converges to a point $x \in \overline{D}$. If $x \in D$, then

$$f(x) = \lim_{\nu\to\infty} f_\nu(w_\nu) = \lim_{\nu\to\infty} w_\nu = x,$$

impossible; therefore $x \in \partial D$.

Now, for every z_0, $z \in D$ we have

$$\lim_{\nu\to\infty} \bigl[k_D(z, w_\nu) - k_D(z_0, w_\nu)\bigr] \le \limsup_{w\to x} \bigl[k_D(z, w) - k_D(z_0, w)\bigr]. \tag{2.4.16}$$

Fix $R>0$ and $z\in E_{z_0}(x,R)$; by (2.4.16), there are $\nu_0\in\mathbf{N}$ and $\varepsilon>0$ such that

$$\forall\nu\ge\nu_0 \qquad k_D(z,w_\nu)<k_D(z_0,w_\nu)+\tfrac{1}{2}\log R-\varepsilon.$$

Since w_ν is a fixed point of $(f_\nu)^k$ for every $k\in\mathbf{N}$, we get

$$\forall\nu\ge\nu_0 \qquad k_D\big((f_\nu)^k(z),w_\nu\big)<k_D(z_0,w_\nu)+\tfrac{1}{2}\log R-\varepsilon.$$

Now,

$$\big|k_D\big((f_\nu)^k(z),w_\nu\big)-k_D\big(f^k(z),w_\nu\big)\big|\le k_D\big((f_\nu)^k(z),f^k(z)\big)\longrightarrow 0$$

as $\nu\to+\infty$; hence there exists $\nu_1\ge\nu_0$ such that

$$\forall\nu\ge\nu_1 \qquad k_D(f^k(z),w_\nu)<k_D(z_0,w_\nu)+\tfrac{1}{2}\log R-\varepsilon/2.$$

Therefore

$$\liminf_{w\to x}\big[k_D(f^k(z),w)-k_D(z_0,w)\big]\le\liminf_{\nu\to\infty}\big[k_D(f^k(z),w_\nu)-k_D(z_0,w_\nu)\big]<\tfrac{1}{2}\log R,$$

and $f^k(z)\in F_{z_0}(x,R)$, **q.e.d.**

In general, the point $x\in\partial D$, whose existence is asserted in Theorem 2.4.19, is not uniquely determined (for instance in the polydisk), and so we shall not speak of a Wolff point associated to the map f. However, we shall later see that x is unique if D is a strongly convex domain.

And now we can proceed to iteration theory in convex domains.

2.4.3 *Convex domains*

This section is devoted to the investigation of the asymptotic behavior of sequences of iterates in bounded convex domains.

In section 2.4.1 we saw that if X is a generic taut manifold, then it is possible to find maps $f\in\mathrm{Hol}(X,X)$ without fixed points and such that $\{f^k\}$ is not compactly divergent. The existence of this kind of maps is a somehow annoying phenomenon: Corollary 2.4.4 is a poor replacement of Corollary 2.4.5, for one would like to study $\Gamma(f)$ without relying too much on the limit manifold of f.

In the ball, these maps do not exist: the sequence $\{f^k\}$ is compactly divergent iff f has no fixed points (Proposition 2.2.30). The next theorem shows that this happens in convex domains too:

Theorem 2.4.20: *Let $D\subset\subset\mathbf{C}^n$ be a bounded convex domain, and $f\in\mathrm{Hol}(D,D)$. Then $\{f^k\}$ is compactly divergent iff f has no fixed points in D.*

Proof: One direction is obvious; conversely, assume that $\{f^k\}$ is not compactly divergent, and let $\rho\colon D\to M$ be the limit retraction. First of all, note that $k_M=k_D|_{M\times M}$. In fact

$$\forall z_1,z_2\in M \quad k_D(z_1,z_2)\le k_M(z_1,z_2)=k_M\big(\rho(z_1),\rho(z_2)\big)\le k_D(z_1,z_2).$$

In particular, a Kobayashi ball in M is nothing but the intersection of a Kobayashi ball of D with M.

Let $\varphi = f|_M$, and denote by Γ the closed subgroup of $\mathrm{Aut}(M)$ generated by φ; we know, by Corollary 2.4.4, that Γ is compact. Take $z_0 \in M$; then the orbit

$$\Gamma(z_0) = \bigl\{\gamma(z_0) \bigm| \gamma \in \Gamma\bigr\}$$

is compact and contained in M. Let

$$\mathcal{C} = \Bigl\{\overline{B_k(w,r)} \Bigm| w \in M,\, r > 0 \quad \text{and} \quad \overline{B_k(w,r)} \supset \Gamma(z_0)\Bigr\},$$

where $B_k(w,r)$ is the Kobayashi ball in D. Every $\overline{B_k(w,r)}$ is compact and convex (by Proposition 2.3.46); therefore, $C = \bigcap \mathcal{C}$ is a not empty compact convex subset of D. We claim that $f(C) \subset C$.

Let $z \in C$; we have to show that $f(z) \in \overline{B_k(w,r)}$ for every $w \in M$ and $r > 0$ such that $\overline{B_k(w,r)} \supset \Gamma(z_0)$. Now, $\overline{B_k(\varphi^{-1}(w),r)} \in \mathcal{C}$: in fact

$$\overline{B_k(\varphi^{-1}(w),r)} \cap M = \varphi^{-1}\bigl(\overline{B_k(w,r)} \cap M\bigr) \supset \varphi^{-1}\bigl(\Gamma(z_0)\bigr) = \Gamma(z_0).$$

Therefore $z \in \overline{B_k(\varphi^{-1}(w),r)}$ and

$$k_D\bigl(w, f(z)\bigr) = k_D\Bigl(f\bigl(\varphi^{-1}(w)\bigr), f(z)\Bigr) \le k_D\bigl(\varphi^{-1}(w), z\bigr) \le r,$$

that is $f(z) \in \overline{B_k(w,r)}$, as we want.

In conclusion, $f(C) \subset C$; by Brouwer's theorem, f must have a fixed point in C, **q.e.d.**

This theorem, besides its importance in iteration theory, is a good tool for the construction of fixed points, as indicated by

Corollary 2.4.21: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and take $f \in \mathrm{Hol}(D,D)$. Let z_0 be an arbitrary point of D; then f has a fixed point iff the sequence $\{f^k(z_0)\}$ has a limit point in D.*

Proof: If $\{f^k(z_0)\}$ has a limit point in D, the sequence $\{f^k\}$ cannot be compactly divergent; hence, by Theorem 2.4.20, f must have a fixed point.

Conversely, assume that f has a fixed point $w \in D$. Then the sequence $\{f^k(z_0)\}$ is contained in the closed Kobayashi ball of center w and radius $k_D(z_0,w)$, which is compact. Hence $\{f^k(z_0)\}$ has a limit point in D, **q.e.d.**

In particular, we can even generalize an argument we used in the proof of Theorem 2.4.20 itself:

Corollary 2.4.22: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and take $f \in \mathrm{Hol}(D, D)$. Assume that there is a compact subset K of D such that $f(K) \subset K$. Then f has a fixed point in D.*

Proof: Apply Corollary 2.4.21 to a point $z_0 \in K$, **q.e.d.**

Now we have sown enough to harvest; in fact, we are ready to prove the announced generalization of the Wolff-Denjoy theorem:

Theorem 2.4.23: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then the sequence $\{f^k\}$ of iterates of f converges to a point $x_0 \in \partial D$.*

Proof: Since f has no fixed points, Theorem 2.4.19 provides us with a point $x \in \partial D$; we claim that $f^k \to x$. Let $h \in \mathrm{Hol}(D, \mathbf{C}^n)$ be a limit point of $\{f^k\}$; if we prove that $h \equiv x$, we are done.

Choose a subsequence $\{f^{k_\nu}\}$ converging to h. By Theorem 2.4.20, $h(D) \subset \partial D$; in particular, h is constant, for D is strongly convex. By Theorem 2.4.19, for every $z_0 \in D$ and $R > 0$ we have

$$\forall \nu \in \mathbf{N} \qquad f^{k_\nu}\big(E_{z_0}(x, R)\big) \subset F_{z_0}(x, R).$$

Taking the limit for $\nu \to +\infty$ we get

$$h\big(E_{z_0}(x, R)\big) \subset \overline{F_{z_0}(x, R)} \cap \partial D = \{x\}$$

(where we are using Theorem 2.4.14), and $h \equiv x$, **q.e.d.**

Therefore Theorem 2.4.23 together with Theorems 2.4.20, 2.4.1 and Corollary 2.4.5 gives a neat description of the asymptotic behavior of a sequence of iterates in a strongly convex domain. Note that, in particular, the point provided by Theorem 2.4.19 in strongly convex domains is unique, for it is the limit of the sequence of $\{f^k\}$.

A careful examination of the proof of Theorem 2.4.23 indicates that we needed the strong convexity only to quote Theorem 2.4.14, showing that the horospheres touch the boundary just in one point. Therefore, if we can somehow generalize Theorem 2.4.14 to the weakly convex case, we can hope in a sort of Wolff-Denjoy theorem for weakly convex domains.

Let $D \subset\subset \mathbf{C}^n$ be a convex C^2 domain. For any $x \in \partial D$, let $\mathbf{n}_x$ be the outer unit normal vector to ∂D at x, and define $\Psi\colon \partial D \times \mathbf{C}^n \to \mathbf{C}$ by

$$\Psi(x, z) = \exp\big[(z - x, \mathbf{n}_x)\big]. \tag{2.4.17}$$

Ψ is the *P-function* of D; it is clear that for any $x \in \partial D$ the function $\Psi_x = \Psi(x, \cdot)$ is a weak peak function for D at x.

The idea is that we can use the P-function to estimate the boundary behavior of the Kobayashi distance in D, exactly as we did in section 2.3.5. To be precise, take $x \in \partial D$. Then the *flat component* $F(x)$ of ∂D at x is the set

$$F(x) = \big\{y \in \partial D \bigm| |\Psi_x(y)| = 1\big\}.$$

In other words, the flat component at x is the intersection of ∂D with the real hyperplane tangent to ∂D at x.

A subset A of ∂D is *saturated* if $F(x) \subset A$ whenever $x \in A$. The flat components form a closed partition of ∂D; since ∂D is compact, it is easy to see that the quotient space with respect to the induced equivalence relation is normal. In particular, two disjoint flat components always have disjoint saturated neighbourhoods, and saturated neighbourhoods form a fundamental system of neighbourhoods of any flat component.

Then we can prove:

Proposition 2.4.24: *Let $D \subset\subset \mathbf{C}^n$ be a bounded convex C^2 domain, and take $x_0 \in \partial D$ and $\delta > 0$. Then there exist ε_0, $\varepsilon_1 \in (0, \delta)$ with $\varepsilon_0 < \varepsilon_1$ such that there is a constant $c \in \mathbf{R}$ so that for all $z \in D \cap B\big(F(x_0), \varepsilon_0\big)$ we have*

$$k_D\big(z, D \setminus B\big(F(x_0), 2\varepsilon_1\big)\big) \geq -\tfrac{1}{2} \log d(z, \partial D) + c.$$

Proof: For any $\varepsilon > 0$ set $U_\varepsilon = \bigcup_{x \in \partial D} P(x, \varepsilon)$. Fix $\varepsilon_1 \in (0, \delta)$ so that $U_{2\varepsilon_1}$ is contained in a tubular neighbourhood of ∂D, and choose a saturated neighbourhood $U \subset\subset B\big(F(x_0), \varepsilon_1\big)$ of $F(x_0)$. Put

$$V_{\varepsilon_1} = \big\{(x, z_0) \in \partial D \times \overline{D} \bigm| x \in \overline{U}, d\big(z_0, F(x_0)\big) \geq \varepsilon_1\big\};$$

note that $(x, z_0) \in V_{\varepsilon_1}$ implies $z_0 \notin \overline{U}$.

Let $\Psi\colon \partial D \times \mathbf{C}^n \to \mathbf{C}$ be the P-function of D. Since U is a saturated neighbourhood and V_{ε_1} is compact, there is $\eta < 1$ such that $|\Psi(x, z_0)| < \eta < 1$ for all $(x, z_0) \in V_{\varepsilon_1}$.

Define $\phi\colon V_{\varepsilon_1} \times \Delta \to \mathbf{C}$ by

$$\phi(x, z_0, \zeta) = \frac{1 - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)} \, \frac{\zeta - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}\zeta},$$

and fix $\gamma \in (\eta, 1)$. If we take a neighbourhood $D_0 \subset\subset \mathbf{C}^n$ of $\overline{D}$ such that $|\Psi(x, z)| < \gamma/\eta$ for all $(x, z) \in \partial D \times \overline{D_0}$, then the map $\Phi(x, z_0, z) = \phi\big(x, z_0, \Psi(x, z)\big)$ is defined and bounded on $V_{\varepsilon_1} \times D_0$. Now choose $\varepsilon_0 \in (0, \varepsilon_1)$ such that $U_{2\varepsilon_0} \subset\subset D_0$ and $B\big(F(x_0), 2\varepsilon_0\big) \subset U$. Then we can proceed exactly as in the proof of Theorem 2.3.54, and the assertion follows, **q.e.d.**

In particular we have

Corollary 2.4.25: *Let $D \subset\subset \mathbf{C}^n$ be a convex C^2 domain; choose two points x_1, $x_2 \in \partial D$ such that $x_1 \notin F(x_2)$ — and hence $x_2 \notin F(x_1)$. Then there exist $\varepsilon_0 > 0$ and $K \in \mathbf{R}$ such that for any $z_1 \in D \cap B\big(F(x_1), \varepsilon_0\big)$ and $z_2 \in D \cap B\big(F(x_2), \varepsilon_0\big)$ we have*

$$k_D(z_1, z_2) \geq -\tfrac{1}{2} \log d(z_1, \partial D) - \tfrac{1}{2} \log d(z_2, \partial D) + K.$$

Proof: Mime the proof of Corollary 2.3.55, **q.e.d.**

Our next aim is then clear:

Proposition 2.4.26: *Let $D \subset\subset \mathbf{C}^n$ be a convex C^2 domain, and $x \in \partial D$. Then for every $z_0 \in D$ and $R > 0$ we have*

$$\overline{F_{z_0}(x,R)} \cap \partial D \subset F(x).$$

Proof: The proof is identical to the second part of the proof of Theorem 2.4.14, replacing Corollary 2.3.55 by Corollary 2.4.25, **q.e.d.**

Summing up, we get a Wolff-Denjoy theorem for weakly convex domains:

Theorem 2.4.27: *Let $D \subset\subset \mathbf{C}^n$ be a convex C^2 domain, and $f \in \mathrm{Hol}(D,D)$ without fixed points. Choose $z_0 \in D$, and let $x_0 \in \overline{D}$ be a limit point of the sequence $\{f^k(z_0)\}$. Then $x_0 \in \partial D$ and*

$$h(D) \subset F(x_0),$$

where h is any limit point of the sequence of iterates of f.

Proof: Let $x \in \partial D$ be provided by Theorem 2.4.19. Exactly as in the proof of Theorem 2.4.23 we see that if h is any limit point of $\{f^k\}$ then

$$h(D) \subset \overline{F_{z_0}(x,R)} \cap \partial D \subset F(x),$$

using Proposition 2.4.26 instead of Theorem 2.4.14. In particular, $x_0 \in F(x)$; hence $F(x) = F(x_0)$, and we are done, **q.e.d.**

So in general, we cannot infer the convergence of the sequence of iterates, but at least we know where the images of limit points of $\{f^k\}$ lie. Sometimes, this can be enough to force the convergence of the whole sequence of iterates, as shown in the following corollary, the last generalization of the Wolff-Denjoy theorem:

Corollary 2.4.28: *Let $D \subset\subset \mathbf{C}^n$ be a convex C^2 domain, and $f \in \mathrm{Hol}(D,D)$. Assume there is $z_0 \in D$ and a point of strong convexity $x_0 \in \partial D$ such that x_0 is a limit point of the sequence $\{f^k(z_0)\}$. Then the whole sequence of iterates of f converges to x_0.*

Proof: Clearly, f has no fixed points. Hence, by Theorem 2.4.27, every limit point of $\{f^k\}$ sends D into $F(x_0)$. But, since x_0 is a point of strong convexity, $F(x_0) = \{x_0\}$ and the assertion follows, **q.e.d.**

Notes

As far as we know, the first paper explicitely devoted to the study of the asymptotic behavior of a sequence of iterates in several variables is Hervé [1951]. He proved close relatives of Theorem 2.4.1 and Corollaries 2.4.4 and 2.4.5 for maps of a bounded taut domain of $\mathbf{C}^2$ into itself. The general versions we presented are in Abate [1988c]. Vesentini [1985] stated a version of Theorem 2.4.1 for holomorphic maps sending a bounded domain of a complex Banach space into itself and having a fixed point.

An embryonal version of Corollaries 2.4.2 and 2.4.5 for domains in $\mathbf{C}^2$ can be found in Lattès [1911].

The idea of the proof of Theorem 2.4.3 comes from Całka [1984], where he described conditions securing in a metric space the boundedness of the orbit of a point under the action of a non-expansive mapping.

Theorem 2.4.6 is due to Bochner and Montgomery [1945, 1947]; Corollary 2.4.7 is in Kobayashi [1970]. Theorem 2.4.8 is due to Remmert [1956, 1957]; Theorem 2.4.9 has been proved by Kaup [1968].

The construction of the horospheres in general domains was originally motivated by Proposition 2.2.20. Actually, there is a very general definition of horospheres in locally compact complete metric spaces which generalizes both our construction and the definition of horospheres in Riemannian geometry. We recall the latter: let M be a complete Riemannian manifold of nonpositive curvature, and $\sigma\colon \mathbf{R} \to M$ a geodesic. The *Busemann function* h associated to σ is defined by

$$\forall z \in M \qquad h(z) = \lim_{t\to\infty} \big[d\big(z, \sigma(t)\big) - t\big],$$

where d is the Riemannian distance on M, and the limit exists for trivial reasons. Then the *horospheres* associated to σ are the sublevels of h, i.e.,

$$E(\sigma, R) = \{z \in M \mid h(z) < R\}. \tag{2.4.18}$$

Now, $t = d(\sigma(0), \sigma(t))$, and so (2.4.18) is akin to our definition.

But the story does not end here: as already announced, there is a more general definition. Let X be a locally compact complete metric space with distance d. We may embed X into $C^0(X)$ mapping $x \in X$ to the function $d_x = d(x, \cdot)$; denote by $\iota(X) \subset C^0(X)$ the image. Now identify two continuous functions on X which differ only by a constant; let $\overline{X}$ be the image of the closure of $\iota(X)$ in $C^0(X)$ under the quotient map π, and set $\partial X = \overline{X} \setminus \pi(\iota(X))$. It is easy to check (using the Ascoli-Arzelà theorem) that $\overline{X}$ and ∂X are compact in the quotient topology, and that $\pi \circ \iota\colon X \to \overline{X}$ is a homeomorphism with the image. ∂X is called the *ideal boundary* of X.

If $h \in \partial X$, then h is a continuous function on X defined up to a constant. Hence the sublevels of h are well defined: they are the *horospheres* at the boundary point h. Now, a preimage $h_0 \in C^0(X)$ of $h \in \partial X$ is the limit of functions of the kind d_{z_ν} for some sequence $\{z_\nu\} \subset X$ without limit points in X. Since every $\pi(d_{z_\nu})$ is defined up to a constant, we can force h_0 to be zero at a fixed point $z_0 \in X$. This amounts to defining the horospheres associated to h by

$$E(h, R) = \big\{z \in X \bigm| \lim_{\nu\to\infty} \big[d(z, z_\nu) - d(z_0, z_\nu)\big] < R\big\}.$$

and this is really quite similar to our approach. A deeper description of this kind of horospheres can be found in Eberlein and O'Neill [1973] and in Ballmann, Gromov and Schroeder [1985]. In this latter book it is also proved that in a complete Riemannian manifold of nonpositive curvature the two definitions of horospheres coincide: the ideal boundary of the manifold is composed only by Busemann functions.

The general properties of horospheres, but the computations in the polydisk, as well as Theorem 2.4.14 are taken from Abate [1988a]; we anticipate that in chapter 2.6 we shall prove that in strongly convex C^3 domains the big and small horopheres coincide. Fadlalla [1973a, b, c], using the Carathéodory distance, introduced something similar to the horospheres described here, and proved a version of Theorem 2.4.14 (Fadlalla [1983]). Another approach to horospheres in bounded symmetric domains is in Bassanelli [1983].

The Julia lemmas Proposition 2.4.15 and Theorem 2.4.16 are in Abate [1988f]; they will be fundamental in chapter 2.7. Another kind of Julia's lemma for bounded domains in $\mathbf{C}^2$ proved using the Bergmann metric is in Wachs [1940]. Theorem 2.4.19 is taken from Abate [1988a].

Theorem 2.4.20 is proved in Abate [1988e]. It would be very interesting to know how much it depends on the convexity of the domain. For instance, Hervé [1954] has shown that if X is a 2-dimensional simply connected taut manifold and $f \in \mathrm{Hol}(X, X)$ is not an automorphism, then $\{f^k\}$ is compactly divergent iff f has no fixed points. In fact, assume $\{f^k\}$ is not compactly divergent, and let M be the limit manifold of f. Since $f \notin \mathrm{Aut}(X)$, the dimension of M is at most 1. If $\dim M = 0$, then f has a fixed point (even attractive: see Corollary 2.4.2). If $\dim M = 1$, then M is a taut simply connected Riemann surface, i.e., a disk, and the assertion follows from the Wolff-Denjoy theorem. So it seems plausible that some sort of generalization of Theorem 2.4.20 should hold for, say, taut manifolds homeomorphic to a ball, but the proof of such a result is at present one of the main open problems in iteration theory. The other one is, of course, the extension of Theorem 2.4.19 to more general domains.

Another proof of Theorem 2.4.20 can be achieved using the notion of aymptotic center introduced by Edelstein [1972]; see Kuczumow and Stachura [1989]. A very preliminary version of Theorem 2.4.20 can also be found in Suzuki [1987].

Theorem 2.4.23 is the main result of Abate [1988a]; the rest of section 2.4.3 comes from Abate [1988e]. Finally, Hervé [1954] is devoted to iteration theory in Δ^2.