Formal Poincaré-Dulac renormalization for holomorphic germs

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Abstract. We describe a general renormalization procedure for germs of holomorphic (or even formal) self-maps, producing a formal normal form simpler than the classical Poincaré-Dulac normal form. As an example of application of our method we provide a complete list of normal forms for quadratic bi-dimensional superattracting germs, that could not be simplified using the classical Poincaré-Dulac normalization only. Finally, we also discuss a few examples of renormalization of germs tangent to the identity, revealing interesting second-order resonance phenomena.

0. Introduction

In the study of a class of holomorphic dynamical systems, an important goal often is the classification under topological, holomorphic or formal conjugation. In particular, for each dynamical system in the class one would like to have a definite way of choosing a (hopefully simpler, possibly unique) representative in the same conjugacy class; a normal form of the original dynamical system.

The most famous kind of normal form for local holomorphic dynamical systems (i.e., germs of holomorphic vector fields at a singular point, or germs of holomorphic self-maps with a fixed point) is the Poincaré-Dulac normal form with respect to formal conjugation, introduced at the end of the nineteenth century. Let us recall very quickly its definition, at least in the setting we are interested here, that is of germs of self-maps with a fixed point, that we can assume to be the origin in ℂⁿ. Moreover, since we are discussing formal normal forms, we shall work with formal transformations of ℂⁿ, that is n-tuples of power series, without discussing here convergence issues.

So let F ∈ ˆOⁿ be a formal transformation in n complex variables, where ˆOⁿ denotes the space of n-tuples of power series in n variables fixing the origin (that is, with vanishing constant term), and let Λ denote the (not necessarily invertible) linear term of F. For simplicity, given a linear map Λ ∈ M_{n,n}(ℂ) we shall denote by ˆOⁿ_{Λ} the set of formal transformations in ˆOⁿ with Λ as linear part. If λ₁, ..., λₙ are the eigenvalues of Λ, we shall say that a multi-index Q of Λ is resonant if there is j ∈ {1, ..., n} such that λ₁^{q₁} · · · λₙ^{qₙ} = λ_j. If this happens, we shall say that the monomial z₁^{q₁} · · · zₙ^{qₙ} e_j is Λ-resonant, where {e₁, ..., eₙ} is the canonical basis of ℂⁿ. Then (for a proof see, e.g., [Ar]):

Theorem 0.1: (Poincaré 1893, Dulac 1904) Let F ∈ ˆOⁿ_{Λ} be a formal transformation in n complex variables fixing the origin, with Λ in Jordan normal form. Then there exists an invertible formal
transformation $\Phi \in \hat{O}_I^n$ with identity linear part such that $G = \Phi^{-1} \circ F \circ \Phi$ contains only $\Lambda$-resonant monomials.

The formal transformation $G$ is a Poincaré-Dulac normal form of $F$; notice that, since $\Phi \in \hat{O}_I^n$, the linear part of $G$ is still $\Lambda$. More generally, we shall say that a $G \in \hat{O}_I^n$ is in Poincaré-Dulac normal form if $G$ contains only $\Lambda$-resonant monomials.

The importance of this result cannot be underestimated, and it has been applied uncountably many times; however it has some limitations. For instance, if $\Lambda = O$ or $\Lambda = I$ then all monomials are resonant; and thus in these cases the Poincaré-Dulac normal form reduces to the original map. More generally, as we shall try and explain below, even when the Poincaré-Dulac normal form is a simplification of the original germ, it is still possible to further simplify the germ (to renormalize it) by applying invertible transformations preserving the property of being in Poincaré-Dulac normal form.

This idea of renormalizing Poincaré-Dulac normal forms is not new in the context of vector fields, where it also has been studied the concept of hypernormal forms, obtained (roughly speaking) by renormalizing infinitely many times; see, e.g., [AFGG, B, BS, G, KOW, LS, Mu1, Mu2] and references therein. On the other hand, this idea has not yet been fully exploited in the context of self-maps (one of the few exceptions is [AT1], where it is applied to a particular class of self-maps with identity linear part). The aim of this paper is to describe in general the renormalization procedure for formal transformations in several complex variables, following the general ideas (but with significantly different details) of the vector field case. We shall then apply this procedure to the case of superattracting (i.e., with $\Lambda = O$) 2-dimensional formal transformations, case that has no analogue in the vector field setting. We shall also discuss a few interesting examples with $\Lambda = I$, in particularly showing the appearance of second-order resonance phenomena.

We conclude this introduction by roughly describing the renormalization procedure. To explain it better, let us first recast the Poincaré-Dulac normalization in slightly different terms (see also [Rü]). For each $\nu \geq 2$ let $\mathcal{H}_\nu$ denote the space of $n$-tuples of homogeneous polynomials in $n$ variables of degree $\nu$. Then every $F \in \hat{O}_I^n$ admits a homogeneous expansion

$$F = \Lambda + \sum_{\nu \geq 2} F_\nu,$$

where $F_\nu \in \mathcal{H}_\nu$ is the $\nu$-homogeneous term of $F$. We shall also use the notation $\{G\}_\nu$ to denote the $\nu$-homogeneous term of a formal transformation $G$.

If

$$\Phi = I + \sum_{\nu \geq 2} H_\nu$$

is the homogeneous expansion of an invertible formal transformation $\Phi \in \hat{O}_I^n$, then it turns out that there exists a linear operator $L_\Lambda: \hat{O}_I^n \to \hat{O}_I^n$, given by

$$L_\Lambda(H) = H \circ \Lambda - \Lambda H,$$

sending each $\mathcal{H}_\nu$ into itself and such that

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) + R_\nu$$

for all $\nu \geq 2$, where $R_\nu$ is a remainder term depending only on $F_\rho$ and $H_\sigma$ with $\rho, \sigma < \nu$. This suggests to consider for each $\nu \geq 2$ splittings of the form

$$\mathcal{H}_\nu = \text{Im} L_\Lambda|_{\mathcal{H}_\nu} \oplus \mathcal{N}_\nu$$

and

$$\mathcal{H}_\nu = \text{Ker} L_\Lambda|_{\mathcal{H}_\nu} \oplus \mathcal{M}_\nu.$$
where $N^\nu$ and $M^\nu$ are suitable complementary subspaces. Then, arguing by induction (see Proposition 2.4), for each $\nu \geq 2$ it is possible to find a unique $H_\nu \in M^\nu$ such that

$$\{ \Phi^{-1} \circ F \circ \Phi \}_\nu = F_\nu - L_\Lambda (H_\nu) + R_\nu \in N^\nu;$$

(0.1)

we can then say that $G = \Phi^{-1} \circ F \circ \Phi$ is in first order normal form with respect to the given choice of complementary subspaces.

When $\Lambda$ is diagonal, $\text{Ker } L_\Lambda$ is generated by the resonant monomials, and $\text{Im } L_\Lambda$ is generated by the non-resonant monomials; in particular, for each $\nu \geq 2$ we have the splitting

$$H^\nu = \text{Im } L_\Lambda|_{H^\nu} \oplus \text{Ker } L_\Lambda|_{H^\nu}.$$

Thus taking $N^\mu = \text{Ker } L_\Lambda|_{H^\nu}$ and $M^\mu = \text{Im } L_\Lambda|_{H^\nu}$ we see that the classical Poincaré-Dulac normal form coincides with the first order normal form with respect to these complementary subspaces (when $\Lambda$ has a nilpotent part the situation is only slightly more complicated; see Section 2 for details).

A consequence of (0.1) is that if $H_\nu \in \text{Ker } L_\Lambda$ then $\{ \Phi^{-1} \circ F \circ \Phi \}_\nu$ does not depend on $H_\nu$; however, $H_\nu$ does affect the remainder terms $R_\rho$ with $\rho > \nu$. In other words, we can use the terms $H_\sigma \in \text{Ker } L_\Lambda$ with $\sigma < \nu$ to simplify the remainder term $R_\nu$.

More precisely, if we take $F$ is in first order normal form, that is

$$F = \Lambda + \sum_{\nu \geq \mu} F_\nu$$

with $F_\nu \in N^\nu$ for all $\nu \geq \mu$ (and $F_\mu \neq 0$), and take $\Phi \in \hat{\Omega}^n$ such that $H_\nu \in \text{Ker } L_\Lambda$ for all $\nu \geq 2$, it turns out that there is an operator $L_{F_\mu, \Lambda}: \hat{\Omega}^n \to \hat{\Omega}^n$ sending each $H^{\nu-\mu+1}$ in $H^\nu$ such that

$$\{ \Phi^{-1} \circ F \circ \Phi \}_\nu = F_\nu - L_{F_\mu, \Lambda}(H_{\nu-\mu+1}) + R'_\nu$$

for all $\nu \geq \mu$, where $R'_\nu$ is a remainder term depending only on $F_\rho$ with $\rho < \nu$ and on $H_\sigma$ with $\sigma < \nu - \mu + 1$ (see Theorem 2.3 for a more complete formula valid without assumptions on $F$ and $\Phi$). The operator $L_{F_\mu, \Lambda}$ is given by

$$L_{F_\mu, \Lambda}(H) = ((\text{Jac } H) \circ \Lambda) \cdot F_\mu - (\text{Jac } F_\mu) \cdot H;$$

notice that, contrarily to $L_\Lambda$, the operator $L_{F_\mu, \Lambda}$ is different from the operators appearing in the renormalization or hypernormalization of singular vector fields, and thus it has to be studied on its own.

If the subspaces $N^\nu$ are chosen (as will be in our case when $\Lambda$ is diagonalizable) so that

$$L_{F_\mu, \Lambda}(\text{Ker } L_\Lambda \cap H^{\nu-\mu+1}) \subseteq N^\nu$$

(0.2)

for all $\nu \geq \mu$ (notice that this condition is particularly easy to state if $N^\nu = \text{Ker } L_\Lambda \cap H^\nu$), then $R'_\nu \in N^\nu$ for all $\nu \geq \mu$. Therefore we can argue as before: putting, for simplicity, $H^\nu = \text{Ker } L_\Lambda \cap H^\nu$, if we choose splittings

$$N^\nu = \text{Im } L_{F_\mu, \Lambda}|_{H^{\nu-\mu+1}} \oplus \hat{N}^\nu$$

and

$$H^\nu = \text{Ker } L_{F_\mu, \Lambda}|_{H^{\nu-\mu+1}} \oplus \hat{N}^\nu$$

$$H^{\nu-\mu+1} = \text{Ker } L_{F_\mu, \Lambda}|_{H^{\nu-\mu+1}} \oplus \hat{N}^{\nu-\mu+1}$$
then arguing again by induction for each $\nu \geq \mu$ it is possible to find a unique $H_{\nu-\mu+1} \in \tilde{\mathcal{M}}^{\nu-\mu + 1}$ such that
\[
\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_{F_\mu, \Lambda}(H_{\nu-\mu+1}) + R'_\nu \in \tilde{\mathcal{N}}^\nu.
\]
We shall then say that $G = \Phi^{-1} \circ F \circ \Phi$ is in renormalized (or second order) Poincaré-Dulac normal form, with respect to the chosen complementary subspaces.

We are left with saying how to choose the complementary subspaces. In this paper, we shall use the orthogonal subspaces with respect to the Fischer Hermitian product, defined by
\[
\langle z_1^{p_1} \cdots z_n^{p_n} e_h, z_1^{q_1} \cdots z_n^{q_n} e_k \rangle = \begin{cases} 
0 & \text{if } h \neq k \text{ or } p_j \neq q_j \text{ for some } j; \\
\frac{p_1! \cdots p_n!}{(p_1 + \cdots + p_n)!} & \text{if } h = k \text{ and } p_j = q_j \text{ for all } j.
\end{cases}
\]

The reason of this choice is that, as we shall see in Sections 3 and 4, it will substantially simplify the expression of the renormalized Poincaré-Dulac normal forms. In particular, when $\Lambda = O$ and $n = 2$, it turns out that, except in a few degenerate cases, the renormalized Poincaré-Dulac normal forms depend on two power series of one variable only.

1. Homogeneous maps

In this section we shall collect a few results on homogeneous polynomials and maps we shall need later.

**Definition 1.1:** We shall denote by $\mathcal{H}^d = (\mathbb{C}_d[z])^n$ the space of *homogenous maps of degree* $d$, i.e., of $n$-tuples of homogeneous polynomials of degree $d \geq 1$ in the variables $(z_1, \ldots, z_n)$. It is well known (see, e.g., [Car, pp. 79–88]) that to each $P \in \mathcal{H}^d$ is associated a unique symmetric multilinear map $\hat{\mathcal{P}}: (\mathbb{C}^n)^d \rightarrow \mathbb{C}^n$ such that
\[
P(z) = \hat{P}(z, \ldots, z)
\]
for all $z \in \mathbb{C}^n$. Notice that $\hat{\mathcal{O}}^n = \prod_{d \geq 1} \mathcal{H}^d$; we set $\mathcal{H} = \prod_{d \geq 2} \mathcal{H}^d$.

Roughly speaking, the symmetric multilinear map associated to a homogeneous map $H$ encodes the derivatives of $H$. For instance, we have

**Lemma 1.1:** If $H \in \mathcal{H}^d$ we have
\[
(\text{Jac } H)(z) \cdot v = d \tilde{H}(v, z, \ldots, z)
\]
for all $z, v \in \mathbb{C}^n$.

**Proof:** For $j = 1, \ldots, n$ and $z \in \mathbb{C}^n$ we have
\[
\frac{H(z + he_j) - H(z)}{h} = \frac{\tilde{H}(z + he_j, \ldots, z + he_j) - \tilde{H}(z, \ldots, z)}{h} = d \tilde{H}(e_j, z, \ldots, z) + O(h),
\]
where $e_j$ is the $j$-th vector of the canonical basis of $\mathbb{C}^n$. Therefore $\frac{\partial H}{\partial z_j}(z) = d \tilde{H}(e_j, z, \ldots, z)$ and
\[
(\text{Jac } H)(z) \cdot v = \sum_{j=1}^n \frac{\partial H}{\partial z_j}(z)v_j = d \sum_{j=1}^n \tilde{H}(e_j, z, \ldots, z)v_j = d \tilde{H}(v, z, \ldots, z).
\]
\[\square\]
Later on we shall need to compute the multilinear map associated to a homogeneous map obtained as a composition. The formula we are interested in is contained in the next lemma.

**Lemma 1.2:** Assume that obtained as a composition. The formula we are interested in is contained in the next lemma.

**Proof:** Write for all and we are done.

We shall say that a homogeneous map \( H \) by \( H \) shall put \( z \) appearing in the classical Poincaré-Dulac theory.

\[ \begin{align*}
\tilde{P}(v, w, \ldots, w) &= \frac{1}{d} \sum_{j=1}^{r} d_j \tilde{K}(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w))
\end{align*} \]

for all \( v, w \in \mathbb{C}^n \).

**Proof:** Write \( z = w + \varepsilon v \). Then

\[ \begin{align*}
P(w) + d\varepsilon \tilde{P}(v, w, \ldots, w) + O(\varepsilon^2)
&= P(w + \varepsilon v) = \tilde{K}(\tilde{H}_{d_1}(w + \varepsilon v, \ldots, w + \varepsilon v), \ldots, \tilde{H}_{d_r}(w + \varepsilon v, \ldots, w + \varepsilon v))
&= \tilde{K}(H_{d_1}(w), \ldots, H_{d_r}(w)) + \varepsilon \sum_{j=1}^{r} d_j \tilde{K}(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)) + O(\varepsilon^2)
\]

and we are done. \( \square \)

**Definition 1.2:** Given a linear map \( \Lambda \in M_{n,n}(\mathbb{C}) \), we define a linear operator \( L_\Lambda: \mathcal{H} \to \mathcal{H} \) by setting

\[ L_\Lambda(H) = H \circ \Lambda - \Lambda H. \]

We shall say that a homogeneous map \( H \in \mathcal{H}^d \) is \( \Lambda \)-resonant if \( L_\Lambda(H) = O \), and we shall denote by \( \mathcal{H}_\Lambda^d = \text{Ker} L_\Lambda \cap \mathcal{H}^d \) the subspace of \( \Lambda \)-resonant homogeneous maps of degree \( d \). Finally, we set \( \mathcal{H}_\Lambda = \prod_{d \geq 2} \mathcal{H}_\Lambda^d \).

When \( \Lambda \) is diagonal, then the \( \Lambda \)-resonant monomials are exactly the resonant monomials appearing in the classical Poincaré-Dulac theory.

**Definition 1.3:** If \( Q = (q_1, \ldots, q_n) \in \mathbb{N}^n \) is a multi-index and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), we shall put \( z^Q = z_1^{q_1} \cdots z_n^{q_n} \). Given \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C}) \), we shall say that \( Q \in \mathbb{N}^n \) with \( q_1 + \cdots + q_n \geq 2 \) is \( \Lambda \)-resonant on the \( j \)-th coordinate if \( \lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j \). We shall denote by \( \text{Res}_j(\Lambda) \) the set of multi-indices \( \Lambda \)-resonant on the \( j \)-th coordinate.

**Remark 1.1:** If \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C}) \) is diagonal, and \( z^Q e_j \in \mathcal{H}^d \) is a homogeneous monomial (with \( q_1 + \cdots + q_n = d \)), then (identifying the matrix \( \Lambda \) with the vector, still denoted by \( \Lambda \), of its diagonal entries) we have

\[ L_\Lambda(z^Q e_j) = (\Lambda^Q - \lambda_j)z^Q e_j. \]

Therefore \( z^Q e_j \) is \( \Lambda \)-resonant if and only if \( Q \) is \( \Lambda \)-resonant in the \( j \)-th coordinate, that is if and only if \( Q \in \text{Res}_j(\Lambda) \). In particular, a basis of \( \mathcal{H}_\Lambda^d \) is given by \( z^Q e_j \) with \( Q \in \text{Res}_j(\Lambda) \) and \( q_1 + \cdots + q_n = d \), and we have

\[ \mathcal{H}^d = \mathcal{H}_\Lambda^d \oplus \text{Im} L_\Lambda|_{\mathcal{H}^d} \]

for all \( d \geq 2 \).

It is possible to detect the \( \Lambda \)-resonance by using the associated multilinear map:
Lemma 1.3: If $\Lambda \in M_{n,n}(\mathbb{C})$ and $H \in \mathcal{H}^d$ then $H$ is $\Lambda$-resonant if and only if

$$\tilde{H}(\Lambda v_1, \ldots, \Lambda v_d) = \Lambda \tilde{H}(v_1, \ldots, v_d)$$

for all $v_1, \ldots, v_d \in \mathbb{C}^n$. In particular, if $H \in \mathcal{H}^d_\Lambda$ then

$$(\text{Jac } H) \circ \Lambda \cdot \Lambda = \Lambda \cdot (\text{Jac } H) .$$

Proof: One direction is trivial. Conversely, assume $H \in \mathcal{H}^d_\Lambda$. By definition, $H$ is $\Lambda$-resonant if and only if $\tilde{H}(\Lambda w, \ldots, \Lambda w) = \Lambda \tilde{H}(w, \ldots, w)$ for all $w \in \mathbb{C}^n$. Put $w = z + \varepsilon v_1$; then

$$\tilde{H}(\Lambda z, \ldots, \Lambda z) + \varepsilon d \tilde{H}(\Lambda v_1, \Lambda z, \ldots, \Lambda z) + O(\varepsilon^2) = \tilde{H}(\Lambda (z + \varepsilon v_1), \ldots, \Lambda (z + \varepsilon v_1))$$

$$= \Lambda \tilde{H}(z + \varepsilon v_1, \ldots, z + \varepsilon v_1)$$

$$= \Lambda \tilde{H}(z, \ldots, z) + \varepsilon d \Lambda \tilde{H}(v_1, z, \ldots, z) + O(\varepsilon^2) ,$$

and thus

$$\tilde{H}(\Lambda v_1, \Lambda z, \ldots, \Lambda z) = \Lambda \tilde{H}(v_1, z, \ldots, z) ;$$

in particular (1.2) is a consequence of Lemma 1.1.

Now put $z = z_1 + \varepsilon v_2$ in (1.3). We get

$$\tilde{H}(\Lambda v_1, \Lambda z_1, \ldots, \Lambda z_1) + \varepsilon (d - 1) \tilde{H}(\Lambda v_1, \Lambda v_2, \Lambda z_1, \ldots, \Lambda z_1) + O(\varepsilon^2)$$

$$= \tilde{H}(\Lambda v_1, \Lambda(z_1 + \varepsilon v_2), \ldots, \Lambda(z_1 + \varepsilon v_2))$$

$$= \Lambda \tilde{H}(v_1, z_1 + \varepsilon v_2, \ldots, z_1 + \varepsilon v_2)$$

$$= \Lambda \tilde{H}(v_1, z_1, \ldots, z_1) + \varepsilon (d - 1) \Lambda \tilde{H}(v_1, v_2, z_1, \ldots, z) + O(\varepsilon^2) ,$$

and hence

$$\tilde{H}(\Lambda v_1, \Lambda v_2, \Lambda z_1, \ldots, \Lambda z_1) = \Lambda \tilde{H}(v_1, v_2, z_1, \ldots, z_1)$$

for all $v_1, v_2, z_1 \in \mathbb{C}^n$. Proceeding in this way we get (1.1). \qed

As we shall see in the next section, the operator $L_\Lambda$ appears in the usual Poincaré-Dulac normalization; for the renormalization we shall need a different operator, that we now introduce.

Definition 1.4: Given $P \in \mathcal{H}^\mu$ and $\Lambda \in M_{n,n}(\mathbb{C})$, let $L_{P,\Lambda}: \mathcal{H}^d \to \mathcal{H}^{d+\mu-1}$ be given by

$$L_{P,\Lambda}(H)(z) = d \tilde{H}(P(z), \Lambda z, \ldots, \Lambda z) - \mu \tilde{P}(H(z), z, \ldots, z) .$$

Remark 1.2: Lemma 1.1 implies that

$$d \tilde{H}(P(z), \Lambda z, \ldots, \Lambda z) = (\text{Jac } H)(\Lambda z) \cdot P(z) .$$

Therefore

$$L_{P,\Lambda}(H) = ((\text{Jac } H) \circ \Lambda) \cdot P - (\text{Jac } P) \cdot H .$$

In particular, when $\Lambda = O$ we have

$$L_{P,O}(H) = -(\text{Jac } P) \cdot H .$$

Remark 1.3: If we take $\mu = 1$ and $P = \Lambda$ we find

$$L_{\Lambda,\Lambda}(H) = d \tilde{H}(\Lambda, \ldots, \Lambda) - \Lambda H = (d - 1)H \circ \Lambda + L_\Lambda(H) ;$$

in particular, $L_{\Lambda,\Lambda} \neq L_\Lambda$.

As mentioned in the introduction, for our machinery to work is important that the operator $L_{P,\Lambda}$ sends the kernel of $L_\Lambda$ into itself. This is the last result of this section:
Corollary 1.4: Take $\Lambda \in M_{n,n}(\mathbb{C})$ and $P \in \mathcal{H}_\Lambda^\mu$. Then $L_{P,\Lambda}(\mathcal{H}_\Lambda^0) \subseteq \mathcal{H}_\Lambda^{d+\mu-1}$ for all $d \geq 2$.

Proof: Using Lemma 1.3 and the definition of $L_{P,\Lambda}$, if $H \in \mathcal{H}_\Lambda^d$ we get

$$L_{P,\Lambda}(H)(\Lambda z) = d \tilde{H}(P(\Lambda z), \Lambda^2 z, \ldots, \Lambda^d z) - \mu \tilde{P}(H(\Lambda z), \Lambda z, \ldots, \Lambda z)$$

$$= d \Lambda \tilde{H}(P(z), \Lambda z, \ldots, \Lambda z) - \mu \Lambda \tilde{P}(H(z), z, \ldots, z)$$

$$= \Lambda L_{P,\Lambda}(H)(z).$$

2. Renormalization

The aim of this section is to describe a procedure associating a renormalized formal Poincaré-Dulac normal form to any germ of holomorphic (or even formal) self-map of $\mathbb{C}^n$ fixing the origin. This is particularly interesting in the case of germs superattracting (i.e., with vanishing linear part) or tangent to the identity (i.e., with identity linear part), because in those cases all monomials are resonant and so the usual Poincaré-Dulac procedure just gives back the original germ.

The idea is the following. Studying the classical proof (see, e.g., [Ar], [R1, 2] and Proposition 2.4 below) of the standard Poincaré-Dulac normalization of a germ $F$, it is clear that we find a unique formal germ $\Psi$ tangent to the identity and containing only nonresonant monomials such that $F_1 = \Psi^{-1} \circ F \circ \Psi$ is in Poincaré-Dulac normal form, that is contains only resonant monomials. Choosing suitable positive definite Hermitian products on the spaces of homogeneous polynomial maps, we shall then be able to determine a unique formal germ $\Phi$ tangent to the identity and containing only resonant monomials such that $\Phi^{-1} \circ F_1 \circ \Phi$ is (in a precise sense) a renormalized Poincaré-Dulac normal form of $F$.

Let us fix a few definitions and notations.

Definition 2.1: We shall denote by $\hat{O}^n$ the space of $n$-tuples of formal power series with vanishing constant term. Furthermore, given $\Lambda \in M_{n,n}(\mathbb{C})$ we shall denote by $\hat{O}^n_\Lambda$ the subset of $F \in \hat{O}^n$ with $dF = \Lambda$.

Definition 2.2: Every $F \in \hat{O}^n$ can be written in a unique way as a formal sum

$$F = \sum_{d \geq 1} F_d$$

with $F_d \in \mathcal{H}^d$; (2.1) is the homogeneous expansion of $F$, and $F_d$ is the $d$-homogeneous term of $F$. We shall often write $\{F\}_d$ for $F_d$. In particular, if $F \in \hat{O}^n_\Lambda$ then $\{F\}_1 = \Lambda$.

The homogeneous terms behave in a predictable way with respect to composition and inverse:

Lemma 2.1: Take $F, G \in \hat{O}^n_\Lambda$, and let $F = \sum_{d \geq 1} F_d$ and $G = \sum_{d \geq 1} G_d$ be their homogeneous expansions. Then

$$\{F \circ G\}_d = \sum_{1 \leq r \leq d \atop d_1 + \cdots + d_r = d} \tilde{F}_r(G_{d_1}, \ldots, G_{d_r})$$

for all $d \geq 1$.

Proof: See, e.g., [LS, Lemma A.4].
Lemma 2.2: Take $\Phi \in \hat{O}_1^n$ with homogeneous expansion $\Phi = I + \sum_{d \geq \delta} H_d$, for some $\delta \geq 2$, and let $\Phi^{-1} = I + \sum_{d \geq 2} K_d$ be the homogeneous expansion of the inverse. Then

$$K_d = -H_d - \sum_{2 \leq r \leq d-1 \atop d_1 + \cdots + d_r = d} \tilde{K}_r(H_{d_1}, \ldots, H_{d_r})$$

(2.2)

for all $d \geq 2$. In particular, $K_2 = -H_2$, and $K_d = 0$ for $d = 2, \ldots, \delta - 1$. Furthermore, if $H_2, \ldots, H_d$ are $\Lambda$-resonant for some $\Lambda \in M_{n,n}(\mathbb{C})$ then also $K_d$ is.

Proof: Lemma 2.1 yields

$$\sum_{1 \leq r \leq d \atop d_1 + \cdots + d_r = d} \tilde{K}_r(H_{d_1}, \ldots, H_{d_r}) = \{\Phi^{-1} \circ \Phi\}_d = O$$

(2.3)

for all $d \geq 2$. Now, when $r = 1$ we necessarily have $d_1 = d$, and so $\tilde{K}_1(H_d) = H_d$ because $K_1 = I$. Analogously, when $r = d$ we necessarily have $d_1 = \cdots = d_r = 1$, and so $K_d(H_1, \ldots, H_1) = K_d$ because $H_1 = I$. Therefore (2.3) becomes

$$H_d + \sum_{2 \leq r \leq d-1 \atop d_1 + \cdots + d_r = d} \tilde{K}_r(H_{d_1}, \ldots, H_{d_r}) + K_d = O,$$

and (2.2) follows. In particular, if $2 \leq d \leq \delta$ in the sum in (2.2) we have $2 \leq d_j < \delta$ (and hence $H_{d_j} = 0$) for at least one $j = 1, \ldots, r$; thus $K_d = -H_d$ for $2 \leq d \leq \delta$, as claimed.

Finally, to prove the last assertion we argue by induction. Assume that $H_2, \ldots, H_d$ are $\Lambda$-resonant. If $d = \delta$ then $K_\delta = -H_\delta$ and thus $K_\delta$ is clearly $\Lambda$-resonant. Assume the assertion true for $d - 1$; in particular, $K_\delta, \ldots, K_{d-1}$ are $\Lambda$-resonant. Then

$$K_d \circ \Lambda = -H_d \circ \Lambda - \sum_{2 \leq r \leq d-1 \atop d_1 + \cdots + d_r = d} \tilde{K}_r(H_{d_1} \circ \Lambda, \ldots, H_{d_r} \circ \Lambda)$$

$$= \Lambda H_d - \sum_{2 \leq r \leq d-1 \atop d_1 + \cdots + d_r = d} \tilde{K}_r(\Lambda H_{d_1}, \ldots, \Lambda H_{d_r}) = \Lambda K_d$$

because $K_2, \ldots, K_{d-1}$ are $\Lambda$-resonant (and we are using Lemma 1.3).

Definition 2.3: Given $\Lambda \in M_{n,n}(\mathbb{C})$, we shall say that $F \in \hat{O}_n$ is $\Lambda$-resonant if $F \circ \Lambda = \Lambda F$. Clearly, $F$ is $\Lambda$-resonant if and only if $\{F\}_d \in H_\Lambda^n$ for all $d \in \mathbb{N}$.

The main technical result of this section is the following:

Theorem 2.3: Given $F \in \hat{O}_3^n$, let $F = \Lambda + \sum_{d \geq \mu} F_d$ be its homogeneous expansion, with $F_\mu \neq O$.

Then for every $\Phi \in \hat{O}_1^n$ with homogeneous expansion $\Phi = I + \sum_{d \geq 2} H_d$ and every $\nu \geq 2$ we have

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) - L_{F_\mu, \Lambda}(H_{\nu - \mu + 1}) + Q_\nu + R_\nu,$$

(2.4)

where $Q_\nu$ depends only on $\Lambda$ and on $H_\gamma$ with $\gamma < \nu$, while $R_\nu$ depends only on $F_\rho$ with $\rho < \nu$ and on $H_\gamma$ with $\gamma < \nu - \mu + 1$, and we put $L_{F_\mu, \Lambda}(H_1) = O$. Furthermore, we have:
(i) if \( H_2, \ldots, H_{\nu-1} \in \mathcal{H}_\Lambda \) then \( Q_\nu = O \); in particular, if \( \Phi \) is \( \Lambda \)-resonant then \( L_\Lambda(H_\nu) = Q_\nu = O \) for all \( \nu \geq 2 \);

(ii) if \( \Phi \) is \( \Lambda \)-resonant then \( \{ \Phi^{-1} \circ F \circ \Phi \}_\nu = O \) for \( 2 \leq \nu < \mu \), \( \{ \Phi^{-1} \circ F \circ \Phi \}_\mu = F_\mu \), and

\[
\{ \Phi^{-1} \circ F \circ \Phi \}_{\mu + 1} = F_{\mu + 1} - L_{F_\mu \Lambda}(H_2);
\]

(iii) if \( F = \Lambda \) then \( R_\nu = O \) for all \( \nu \geq 2 \);

(iv) if \( F_2, \ldots, F_{\nu-1} \) and \( H_2, \ldots, H_{\nu-\mu} \) are \( \Lambda \)-resonant then \( R_\nu \) is \( \Lambda \)-resonant.

**Proof:** Using twice Lemma 2.1 we get

\[
\{ \Phi^{-1} \circ F \circ \Phi \}_\nu = \sum_{\nu_1 + \ldots + \nu_s = \nu} \tilde{K}_s(\{ F \circ \Phi \}_{\nu_1}, \ldots, \{ F \circ \Phi \}_{\nu_s})
\]

\[
= \sum_{\nu_1 + \ldots + \nu_s = \nu} \sum_{1 \leq r_1 \leq \nu} \ldots \sum_{1 \leq r_s \leq \nu} \tilde{K}_s(\tilde{F}_{r_1}(H_{d_{11}}, \ldots, H_{d_{1r_1}}), \ldots, \tilde{F}_{r_s}(H_{d_{s1}}, \ldots, H_{d_{sr_s}}))
\]

\[
= T_\nu + S_1(\nu) + \sum_{s \geq 2} S_s(\nu),
\]

where \( \Phi^{-1} = I + \sum_{d \geq 2} K_d \) is the homogeneous expansion of \( \Phi^{-1} \), and:

(1) \[
T_\nu = \sum_{1 \leq r_1 \leq \nu} \tilde{K}_s(\Lambda H_{\nu_1}, \ldots, \Lambda H_{\nu_s})
\]

is obtained considering only the terms with \( r_1 = \ldots = r_s = 1 \);

(2) \[
S_1(\nu) = \sum_{d_1 + \ldots + d_{\nu} = \nu} \tilde{F}(H_{d_1}, \ldots, H_{d_{\nu}})
\]

contains the terms with \( s = 1 \) and \( r_1 > 1 \); and

(3) \[
S_s(\nu) = \sum_{\nu_1 + \ldots + \nu_s = \nu} \sum_{1 \leq r_1 \leq \nu_1} \ldots \sum_{1 \leq r_s \leq \nu_s} \tilde{K}_s(\tilde{F}_{r_1}(H_{d_{11}}, \ldots, H_{d_{1r_1}}), \ldots, \tilde{F}_{r_s}(H_{d_{s1}}, \ldots, H_{d_{sr_s}}))
\]

contains the terms with fixed \( s \geq 2 \) and at least one \( r_j \) greater than 1 (and thus greater than or equal to \( \mu \), because \( F_2 = \ldots = F_{\nu-1} = O \) by assumption).

Let us first study \( T_\nu \). The summand corresponding to \( s = 1 \) is \( \Lambda H_\nu \); the summand corresponding to \( s = \nu \) is \( K_\nu \circ \Lambda \); therefore

\[
T_\nu = \Lambda H_\nu + K_\nu \circ \Lambda + \sum_{2 \leq s \leq \nu-1} \sum_{\nu_1 + \ldots + \nu_s = \nu} \tilde{K}_s(\Lambda H_{\nu_1}, \ldots, \Lambda H_{\nu_s}) = -L_\Lambda(H_\nu) + Q_\nu,
\]

where, using Lemma 2.2 to express \( K_\nu \),

\[
Q_\nu = \sum_{2 \leq s \leq \nu-1} \left[ \tilde{K}_s(\Lambda H_{\nu_1}, \ldots, \Lambda H_{\nu_s}) - \tilde{K}_s(H_{\nu_1} \circ \Lambda, \ldots, H_{\nu_s} \circ \Lambda) \right]
\]
depends only on $\Lambda$ and $H_\gamma$ with $\gamma < \nu$ because $2 \leq s \leq \nu - 1$ in the sum. In particular, if $H_1, \ldots, H_{\nu-1} \in \mathcal{H}_\Lambda$ then $Q_\mu = O$, and (i) is proved.

Now let us study $S_1(\nu)$. First of all, we clearly have $S_1(\nu) = O$ for $2 \leq \nu < \mu$, and $S_1(\mu) = F_\mu$. When $\nu > \mu$ we can write

\[
S_1(\nu) = F_\nu + \sum_{\mu \leq \nu \leq \nu - 1} \tilde{F}_\nu(H_{d_1}, \ldots, H_{d_r})
\]

\[
= F_\nu + \mu \tilde{F}_\mu(H_{\nu - \mu + 1}, \ldots, I) + \sum_{d_1 + \cdots + d_j = \nu, \mu + 1 \leq \nu} \tilde{F}_\mu(H_{d_1}, \ldots, H_{d_s}) + \sum_{\mu + 1 \leq \nu \leq \nu - 1} \tilde{F}_\nu(H_{d_1}, \ldots, H_{d_r}) .
\]

in particular, $S_1(\mu + 1) = F_{\mu + 1} + \mu \tilde{F}_\mu(H_2, I, \ldots, I)$. Notice that the two remaining sums depend only on $F_\rho$ with $\rho < \nu$ and on $H_\nu$ with $\gamma < \nu - \mu + 1$ (in the first sum is clear; for the second one, if $d_j \geq \nu - \mu + 1$ for some $j$ we then would have $d_1 + \cdots + d_r \geq \nu - \mu + r - 1 \geq \nu + 1$, impossible). Summing up we have

\[
S_1(\nu) = \begin{cases} 
O & \text{for } 2 \leq \nu < \mu, \\
F_\mu & \text{for } \nu = \mu, \\
F_{\mu + 1} + \mu \tilde{F}_\mu(H_2, I, \ldots, I) & \text{for } \nu = \mu + 1, \\
F_\nu + \mu \tilde{F}_\mu(H_{\nu - \mu + 1}, I, \ldots, I) + R_\nu^1 & \text{for } \nu > \mu + 1,
\end{cases}
\]

where

\[
R_\nu^1 = \sum_{\max(d_j) = \nu, d_1 + \cdots + d_s = \nu} \tilde{F}_\mu(H_{d_1}, \ldots, H_{d_s}) + \sum_{\mu + 1 \leq \nu \leq \nu - 1} \tilde{F}_\nu(H_{d_1}, \ldots, H_{d_r})
\]

depends only on $F_\rho$ with $\rho < \nu$ and on $H_\nu$ with $\gamma < \nu - \mu + 1$.

Let us now discuss $S_s(\nu)$ for $s \geq 2$. First of all, the condition $\max\{r_1, \ldots, r_s\} \geq \mu$ implies

\[
\mu + s - 1 \leq r_1 + \cdots + r_s \leq \nu_1 + \cdots + \nu_s = \nu,
\]

that is $s \leq \nu - \mu + 1$. In particular, $S_s(\nu) = O$ if $\nu \leq \mu$ or if $s > \nu - \mu + 1$. Moreover, if we had $d_{ij} \geq \nu - \mu + 1$ for some $1 \leq i \leq s$ and $1 \leq j \leq r_s$ we would get

\[
\nu = d_{11} + \cdots + d_{sr_s} \geq \nu - \mu + 1 + r_1 + \cdots + r_s - 1 \geq \nu - \mu + 1 + \mu + s - 1 - 1 = \nu + s - 1 > \nu,
\]

impossible. This means that $S_s(\nu)$ depends only on $F_\rho$ with $\rho < \nu$ for all $s$, on $H_\gamma$ with $\gamma < \nu - \mu + 1$ when $s < \nu - \mu + 1$, and that $S_{\nu - \mu + 1}(\nu)$ depends on $H_{\nu - \mu + 1}$ only because it contains $K_{\nu - \mu + 1}$. Furthermore, the conditions $\max\{r_1, \ldots, r_{\nu - \mu + 1}\} \geq \mu$ and $\nu_1 + \cdots + \nu_{\nu - \mu + 1} = \nu$ imply that

\[
S_{\nu - \mu + 1}(\nu) = (\nu - \mu + 1)\tilde{K}_{\nu - \mu + 1}(F_\mu, \Lambda, \ldots, \Lambda) = -(\nu - \mu + 1)\tilde{H}_{\nu - \mu + 1}(F_\mu, \Lambda, \ldots, \Lambda) + R_{\nu}^2,
\]

where (using Lemmas 1.2 and 2.2)

\[
R_{\nu}^2 = \sum_{2 \leq r \leq \nu - \mu} \sum_{d_1 + \cdots + d_r = \nu - \mu + 1} d_j \tilde{K}_r(H_{d_1} \circ \Lambda, \ldots, \tilde{H}_{d_j}(F_\mu, \Lambda, \ldots, \Lambda), \ldots, H_{d_r} \circ \Lambda)
\]

depends only on $\Lambda, F_\mu$ and $H_\gamma$ with $\gamma < \nu - \mu + 1$. 

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Putting everything together, we have
\[
\{\Phi^{-1} \circ F \circ \Phi\}_\nu = T_\nu + S_1(\nu) + \sum_{s=2}^{\nu-\mu+1} S_s(\nu)
\]
\[
= F_\nu - L_\Lambda(H_\nu) + Q_\nu + \begin{cases}
O & \text{if } 2 \leq \nu \leq \mu, \\
-L_{F_\nu \Lambda}(H_2) & \text{if } \nu = \mu + 1, \\
-L_{F_\nu \Lambda}(H_{\nu-\mu+1}) + R_\nu & \text{if } \nu > \mu + 1,
\end{cases}
\]
where
\[
R_\nu = R^1_\nu + R^2_\nu + \sum_{s=2}^{\nu-\mu} S_s(\nu)
\]
depends only on \(F\) with \(\rho < \mu\) and on \(H_\gamma\) with \(\gamma < \nu - \mu + 1\). In particular, if \(F = \Lambda\) then we have \(S_s(\nu) = 0\) for all \(s \geq 1\) and hence \(R_\nu = 0\) for all \(\nu \geq 2\).

In this way we have proved (2.4) and parts (i), (ii) and (iii). Concerning (iv), it suffices to notice that if \(F_2, \ldots, F_{\nu-1}\) and \(H_2, \ldots, H_{\nu-\mu+1}\) are \(\Lambda\)-resonant, then also \(R^1_\nu, S_2(\nu), \ldots, S_{\nu-\mu}(\nu)\) and \(R^2_\nu\) (by Lemmas 1.3 and 2.2) are \(\Lambda\)-resonant.

We can now prove the existence of a first order normalization in the sense described in the introduction.

**Proposition 2.4:** Take \(\Lambda \in M_{n,n}(\mathbb{C})\) and for each \(\nu \geq 2\) choose two subspaces \(N^\nu, M^\nu \subseteq \mathcal{H}^\nu\) such that \(\mathcal{H}^\nu = \text{Im } L_\Lambda|_{\mathcal{H}^\nu} \oplus N^\nu\) and \(\mathcal{H}^\nu = \mathcal{H}^\nu_{\Lambda} \oplus M^\nu\). Then for every \(F \in \hat{O}_\Lambda\) there exists a unique \(\Phi = I + \sum_{d \geq 2} H_d\) such that \(H_d \in M^d\) for all \(d \geq 2\) and \(\{\Phi^{-1} \circ F \circ \Phi\}_\nu \in N^\nu\) for all \(\nu \geq 2\).

**Proof:** Notice that, by construction, \(L_\Lambda(\mathcal{H}^\nu) = L_\Lambda(M^\nu)\) and \(L_\Lambda|_{M^\nu}\) is injective. Now put \(G = \Phi^{-1} \circ F \circ \Phi;\) we define \(H_d\) by induction. For \(d = 2\), we see that there exist a unique \(G \in N^2\) and a unique \(H \in M^2\) such that \(F_2 = G + L_\Lambda(H)\). Since (2.4) says that
\[
G_2 = F_2 - L_\Lambda(\{\Phi\}_2) = G + L_\Lambda(H) - L_\Lambda(\{\Phi\}_2),
\]
to get \(G_2 \in N^2\) with \(\{\Phi\}_2 \in M^2\) we must take \(\{\Phi\}_2 = H\).

Assume now that we have defined \(H_j \in M^j\) for \(j = 2, \ldots, H_{d-1}\). In particular, this determines completely the terms \(Q_d, R_d\) and \(L_{F_j \Lambda}(H_{d-1})\) in (2.4). So there exist a unique \(G \in N^d\) and a unique \(H \in M^d\) such that \(F_d - L_{F_j \Lambda}(H_{d-1}) + Q_d + R_d = G + L_\Lambda(H)\). Then to get \(G_d \in N^d\) with \(\{\Phi\}_d \in M^d\) the only choice is \(\{\Phi\}_d = H\), and thus \(G_d = G\).

There are a few natural choices for the subspaces \(N^\nu\) and \(M^\nu\) (see, e.g., [Mu1, Chapter 4]). If \(\Lambda\) is diagonal, then Remark 1.1 shows that we can take \(N^\nu = \mathcal{H}^\nu_{\Lambda}\) and \(M^\nu = \text{Im } L_\Lambda|_{\mathcal{H}^\nu}\), and thus Proposition 2.4 gives nothing but the usual Poincaré-Dulac normal form.

Another possibility arises choosing on each \(\mathcal{H}^\nu\) a positive definite Hermitian product. Then, denoting by \(L^*_\Lambda\) the adjoint operator of \(L_\Lambda\), we have
\[
\mathcal{H}^\nu = \text{Im } L_\Lambda|_{\mathcal{H}^\nu} \oplus \text{Ker } L^*_\Lambda|_{\mathcal{H}^\nu} = \mathcal{H}^\nu_{\Lambda} \oplus \text{Im } L^*_\Lambda|_{\mathcal{H}^\nu},
\]
and thus we can take \(N^\nu = \text{Ker } L^*_\Lambda|_{\mathcal{H}^\nu}\) and \(M^\nu = \text{Im } L^*_\Lambda|_{\mathcal{H}^\nu}\).

If we use the Fischer Hermitian product introduced in (0.3), it turns out that \(L^*_\Lambda = L_{\Lambda^*}\), where \(\Lambda^*\) is the matrix adjoint of \(\Lambda\) (see, e.g., [Mu1, Lemma 4.6.6]). Furthermore, when \(\Lambda\) is diagonal we clearly have \(\text{Ker } L_{\Lambda^*} = \text{Ker } L_\Lambda\), and thus we have again recovered the usual Poincaré-Dulac normal form. More generally, if \(\Lambda = D + N\) is in Jordan normal form, with \(D\) diagonal and \(N\) nilpotent, then \(\text{Ker } L_\Lambda = \text{Ker } L_D \cap \text{Ker } L_{N^\nu} \subseteq \text{Ker } L_D\) (see, e.g., [Mu1, Lemma 4.6.9]), and thus in this case too we have recovered the usual Poincaré-Dulac normal form (composed by monomials resonant with respect to the eigenvalues of \(\Lambda\), i.e., \(D\)-resonant).

We can now introduce the renormalized Poincaré-Dulac normal form.
Definition 2.4: Given $\Lambda \in M_{n,n}(\mathbb{C})$, we shall say that $G \in \hat{O}_\Lambda^R$ is in renormalized Poincaré-Dulac normal form if $G = \Lambda$ or the homogeneous expansion $G = \Lambda + \sum_{d \geq \mu} G_d$ of $G$ satisfies the following conditions:

(a) $G_\mu \in \mathcal{H}_\Lambda^\mu \setminus \{0\}$;
(b) $G_d \in \mathcal{H}_\Lambda^{\mu} \cap (\operatorname{Im} L_{G_{\mu},\Lambda})^\perp$ for all $d > \mu$ (where we are using the Fischer Hermitian product).

Given $F \in \hat{O}_\Lambda^R$, we shall say that $G \in \hat{O}_\Lambda^R$ is a renormalized Poincaré-Dulac normal form of $F$ if $G$ is in renormalized Poincaré-Dulac normal form and $G = \Phi^{-1} \circ F \circ \Phi$ for some $\Phi \in \hat{O}_\Lambda^R$.

To proceed with the renormalization as explained in the introduction, we need condition (0.2), that is we need to check that the operator $L_{F_{\mu},\Lambda}$ with $F_{\mu} \in \mathcal{N}_\mu^\mu$ sends $\mathcal{H}_\Lambda^{\mu-\mu+1}$ into $\mathcal{N}_\nu^\nu$. When $\Lambda$ is diagonal, we have $\mathcal{N}_\nu^\nu = \mathcal{H}_\Lambda^\nu$ for all $\nu \geq 2$, and hence (0.2) follows from Corollary 1.4. We then have the renormalized normal form we were looking for:

**Theorem 2.5:** Let $\Lambda \in M_{n,n}(\mathbb{C})$ be diagonal. Then each $F \in \hat{O}_\Lambda^R$ admits a renormalized Poincaré-Dulac normal form. More precisely, if $F = \Lambda + \sum_{d \geq \mu} F_d$ is in Poincaré-Dulac normal form (and $F \neq \Lambda$) then there exists a unique $\Lambda$-resonant $\Phi = I + \sum_{d \geq 2} H_d \in \hat{O}_\Lambda^R$ such that $H_d \in (\operatorname{Ker} L_{F_{\mu},\Lambda})^\perp$ for all $d \geq 2$ and $G = \Phi^{-1} \circ F \circ \Phi$ is in renormalized Poincaré-Dulac normal form.

**Proof:** By Proposition 2.4 we can assume that $F$ is in Poincaré-Dulac normal form. If $F \equiv \Lambda$ we are done; assume then that $F \neq \Lambda$.

First of all, by Proposition 2.3 if $\Phi$ is $\Lambda$-resonant we have $\{\Phi^{-1} \circ F \circ \Phi\}_d = F_d$ for all $d \leq \mu$.

In particular, $F_{\mu} \in \mathcal{H}_\Lambda^\mu$; therefore, by Corollary 1.4, $\operatorname{Im} L_{F_{\mu},\Lambda}|_{\mathcal{H}_\Lambda^{\mu-\mu+1}} \subseteq \mathcal{H}_\Lambda^\mu$. We then have the splittings

$$\mathcal{H}_\Lambda^{\mu} = \operatorname{Im} L_{F_{\mu},\Lambda}|_{\mathcal{H}_\Lambda^{\mu-\mu+1}} \oplus (\operatorname{Im} L_{F_{\mu},\Lambda}|_{\mathcal{H}_\Lambda^{-\mu+1}})^\perp,$$

and

$$\mathcal{H}_\Lambda^{\mu-\mu+1} = \operatorname{Ker} L_{F_{\mu},\Lambda}|_{\mathcal{H}_\Lambda^{\mu-\mu+1}} \oplus (\operatorname{Ker} L_{F_{\mu},\Lambda}|_{\mathcal{H}_\Lambda^{-\mu+1}})^\perp.$$

Hence we can find a unique $G \in (\operatorname{Im} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^{\mu+1}$ and a unique $H \in (\operatorname{Ker} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^2$ such that $F_{\mu+1} = G + L_{F_{\mu},\Lambda}(H)$. Then Proposition 2.3 yields

$$\{\Phi^{-1} \circ F \circ \Phi\}_{\mu+1} = F_{\mu+1} - L_{F_{\mu},\Lambda}(\{\Phi\}_2) = G + L_{F_{\mu},\Lambda}(H) - L_{F_{\mu},\Lambda}(\{\Phi\}_2);$$

so to get $\{\Phi^{-1} \circ F \circ \Phi\}_{\mu+1} \in (\operatorname{Im} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^{\mu+1}$ with $\{\Phi\}_2 \in (\operatorname{Ker} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^2$, we must necessarily take $\{\Phi\}_2 = H$.

Assume, by induction, that we have uniquely determined $H_2, \ldots, H_{d-1} \in (\operatorname{Im} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda$; in particular, this determines completely $R_d \in \mathcal{H}_\Lambda^2$ in (2.4). Hence there exist a unique $G$ in $(\operatorname{Im} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^d$ and a unique $H \in (\operatorname{Ker} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^{d-\mu+1}$ such that $F_d + R_d = G + L_{F_{\mu},\Lambda}(H)$. So to get $\{\Phi^{-1} \circ F \circ \Phi\}_d \in (\operatorname{Im} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^d$ with $\{\Phi\}_d \in (\operatorname{Ker} L_{F_{\mu},\Lambda})^\perp \cap \mathcal{H}_\Lambda^{d-\mu+1}$ the only choice is $\{\Phi\}_d = H$, and thus $\{\Phi^{-1} \circ F \circ \Phi\}_d = G$. \(\square\)

As examples of applications of this method, in the remaining two sections we shall study cases where the usual Poincaré-Dulac normal form reduces to the original map.

3. Examples with $\Lambda = O$

In this section we shall completely describe the renormalized normal forms obtained when $n = \mu = 2$ and $\Lambda = O$, that is in the 2-dimensional quadratic superattracting case. It is worthwhile to remark
that, except in a few degenerate instances, the normal form will be expressed just in terms of two power series of one variable, and thus we shall obtain a drastic simplification of the germs.

In [A3] we showed that, up to a linear change of variable, we can assume that the quadratic term $F_2$ is of one (and only one) of the following forms:

\[
(\infty) \quad F_2(z, w) = (z^2, zw);
\]

\[
(1_{00}) \quad F_2(z, w) = (0, -z^2);
\]

\[
(1_{10}) \quad F_2(z, w) = (-z^2, -(z^2 + zw));
\]

\[
(1_{11}) \quad F_2(z, w) = (-zw, -(z^2 + w^2));
\]

\[
(2_{00}) \quad F_2(z, w) = (0, zw);
\]

\[
(2_{01}) \quad F_2(z, w) = (zw, zw + w^2);
\]

\[
(2_{10}) \quad F_2(z, w) = (-\rho z^2, (1 - \rho)zw), \text{ with } \rho \neq 0;
\]

\[
(2_{11}) \quad F_2(z, w) = (\rho z^2 + zw, (1 + \rho)zw + w^2), \text{ with } \rho \neq 0;
\]

\[
(3_{00}) \quad F_2(z, w) = (z^2 - zw, 0);
\]

\[
(3_{10}) \quad F_2(z, w) = (\rho(z^2 + zw), (1 - \rho)(zw - w^2)), \text{ with } \rho \neq 0, 1;
\]

\[
(3_{11}) \quad F_2(z, w) = (-\rho z^2 + (1 - \tau)zw, (1 - \rho)zw - \tau w^2), \text{ with } \rho, \tau \neq 0 \text{ and } \rho + \tau \neq 1
\]

(where the symbols refer to the number of characteristic directions and to their indices; see also [AT2]).

We shall use the standard basis \(\{u_{d,j}, v_{d,j}\}_{j=0,\ldots,d}\) of \(\mathcal{H}^d\), where

\[
u_{d,j} = (z^j w^{d-j}, 0) \quad \text{and} \quad v_{d,j} = (0, z^j w^{d-j}),
\]

and we shall endow \(\mathcal{H}^d\) with the usual Fischer scalar product, so that \(\{u_{d,j}, v_{d,j}\}_{j=0,\ldots,d}\) is an orthogonal basis and

\[
\|u_{d,j}\|^2 = \|v_{d,j}\|^2 = \left(\frac{d}{j}\right)^{-1}.
\]

Finally, we recall that when \(\Lambda = O\) the operator \(L = L_{F_2,\Lambda}\) is given by

\[
L(H) = -\text{Jac}(F_2) \cdot H.
\]

We shall now study separately each case.

- **Case (\(\infty\)).**

In this case we have

\[
L(u_{d,j}) = -2u_{d+1,j+1} - v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = -v_{d+1,j+1}
\]

for all \(d \geq 2\) and \(j = 0, \ldots, d\). Therefore

\[
\text{Im } L|_{\mathcal{H}^d} = \text{Span } (u_{d+1,2}, \ldots, u_{d+1,d+1}, 2u_{d+1,1} + v_{d+1,0}, v_{d+1,1}, \ldots, v_{d+1,d+1})
\]

and thus

\[
(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span } (u_{d+1,0}, (d + 1)u_{d+1,1} - 2v_{d+1,0}).
\]

It then follows that every formal power series of the form

\[
F(z, w) = (z^2 + O_3, zw + O_3)
\]

(where \(O_3\) denotes a remainder term of order at least 3) is formally conjugated to a power series of the form

\[
G(z, w) = (z^2 + \varphi(w) + zw, zw - 2\psi(w))
\]

where \(\varphi, \psi \in \mathbb{C}[[\zeta]]\) are arbitrary power series of order at least 3. Notice that (here and in later formulas) the appearance of the derivative (which simplifies the expression of the normal form) is due to the fact we are using the Fischer Hermitian product; using another Hermitian product might lead to more complicated normal forms.
• Case (100).
  In this case we have
  \[ L(u_{d,j}) = 2v_{d+1,j+1} \quad \text{and} \quad L(v_{d,j}) = 0 \]
  for all \( d \geq 2 \) and \( j = 0, \ldots, d \). Therefore
  \[ \text{Im} \, L|_{\mathcal{H}^d} = \text{Span} \left( v_{d+1,1}, \ldots, v_{d+1,d+1} \right) , \]
  and thus
  \[ (\text{Im} \, L|_{\mathcal{H}^d})^\perp = \text{Span} \left( u_{d+1,0}, \ldots, u_{d+1,d+1}, v_{d+1,0} \right) . \]
  It then follows that every formal power series of the form
  \[ F(z, w) = (O_3, -z^2 + O_3) \]
  is formally conjugated to a power series of the form
  \[ G(z, w) = (\Phi(z, w), -z^2 + \psi(w)) \]
  where \( \psi \in \mathbb{C}[\zeta] \) and \( \Phi \in \mathbb{C}[z, w] \) are arbitrary power series of order at least 3.

• Case (110).
  In this case we have
  \[ L(u_{d,j}) = 2u_{d+1,j+1} + 2v_{d+1,j+1} + v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = v_{d+1,j+1} \]
  for all \( d \geq 2 \) and \( j = 0, \ldots, d \). Therefore
  \[ \text{Im} \, L|_{\mathcal{H}^d} = \text{Span} \left( 2u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1} \right) , \]
  and thus
  \[ (\text{Im} \, L|_{\mathcal{H}^d})^\perp = \text{Span} \left( (d+1)u_{d+1,1} - 2v_{d+1,0} \right) . \]
  It then follows that every formal power series of the form
  \[ F(z, w) = (-z^2 + O_3, -z^2 - zw + O_3) \]
  is formally conjugated to a power series of the form
  \[ G(z, w) = \left( -z^2 + \varphi(w) + zw', -z^2 - zw - \psi(w) \right) \]
  where \( \varphi, \psi \in \mathbb{C}[\zeta] \) are arbitrary power series of order at least 3.

• Case (111).
  In this case we have
  \[ L(u_{d,j}) = u_{d+1,j} + 2v_{d+1,j+1} \quad \text{and} \quad L(v_{d,j}) = u_{d+1,j+1} + 2v_{d+1,j} \]
  for all \( d \geq 2 \) and \( j = 0, \ldots, d \). It follows that
  \[ \text{Im} \, L|_{\mathcal{H}^d} = \text{Span} \left( u_{d+1,0} - u_{d+1,2}, \ldots, u_{d+1,d-1} - u_{d+1,d+1}, \right. \]
  \[ v_{d+1,2} - v_{d+1,0}, \ldots, v_{d+1,d+1} - v_{d+1,d-1}, u_{d+1,0} + 2v_{d+1,1}, u_{d+1,1} + 2v_{d+1,0} \), \]
and a few computations yield

$$(\text{Im } L)|_{\mathcal{H}^d} \perp = \text{Span} \left( \sum_{j=0}^{d+1} \binom{d+1}{j} \left( v_{d+1,j} - 2u_{d+1,j} \right), \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \left( v_{d+1,j} + 2u_{d+1,j} \right) \right).$$

$$= \text{Span} \left( (-2(z+w)^{d+1}, (z+w)^{d+1}), (2(w-z)^{d+1}, (w-z)^{d+1}) \right)$$

It then follows that every formal germ of the form

$$F(z, w) = (-zw + O_3, -z^2 - w^2 + O_3)$$

is formally conjugated to a germ of the form

$$G(z, w) = (-zw - 2\varphi(z+w) + 2\psi(w-z), -z^2 - w^2 + \varphi(z + w) + \psi(w-z))$$

where $\varphi, \psi \in \mathbb{C}[\zeta]$ are arbitrary power series of order at least 3. Again, the fact that the normal form is expressed in terms of power series evaluated in $z + w$ and $z - w$ due to the fact we are using the Fischer Hermitian product.

- **Case (2001).**

In this case we have

$$L(u_{d,j}) = -v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = -v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\text{Im } L|_{\mathcal{H}^d} = \text{Span} (v_{d+1,0}, \ldots, v_{d+1,d+1})$$

and hence

$$(\text{Im } L)|_{\mathcal{H}^d} \perp = \text{Span} (u_{d+1,0}, \ldots, u_{d+1,d+1}).$$

It then follows that every formal germ of the form

$$F(z, w) = (O_3, zw + O_3)$$

is formally conjugated to a germ of the form

$$G(z, w) = (\Phi(z, w), zw)$$

where $\Phi \in \mathbb{C}[z, w]$ is a power series of order at least three.

- **Case (2011).**

In this case we have

$$L(u_{d,j}) = -u_{d+1,j} - v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = -u_{d+1,j+1} - 2v_{d+1,j} - v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\text{Im } L|_{\mathcal{H}^d} = \text{Span} (u_{d+1,0}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d+1}, u_{d+1,d}, u_{d+1,d+1} + v_{d+1,d+1} + 2v_{d+1,d})$$

and hence

$$(\text{Im } L)|_{\mathcal{H}^d} \perp = \text{Span} \left( (d+1)u_{d+1,d} - (d+1)v_{d+1,d} + 2v_{d+1,d+1}, u_{d+1,d+1} - v_{d+1,d+1} \right).$$

It then follows that every formal germ of the form

$$F(z, w) = (zw + O_3, zw + w^2 + O_3)$$

is formally conjugated to a germ of the form

$$G(z, w) = (zw + w\varphi'(z) + \psi(z), zw + w^2 + 2\varphi(z) - w\varphi'(z) - \psi(z)),$$

where $\varphi, \psi \in \mathbb{C}[\zeta]$ are arbitrary power series of order at least 3.
• Case \((2_{10\rho})\).
In this case we have
\[
L(u_{d,j}) = 2\rho u_{d+1,j+1} + (\rho - 1)v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = (\rho - 1)v_{d+1,j+1}
\]
for all \(d \geq 2\) and \(j = 0, \ldots, d\). We clearly have two subcases to consider.

If \(\rho = 1\) then
\[
\text{Im } L|_{\mathcal{H}^d} = \text{Span} \left( u_{d+1,1}, \ldots, u_{d+1,d+1} \right),
\]
and hence
\[
(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span} \left( u_{d+1,0}, v_{d+1,0}, \ldots, v_{d+1,d+1} \right).
\]

It then follows that every formal germ of the form
\[
F(z, w) = (-z^2 + O_3, O_3)
\]
is formally conjugated to a germ of the form
\[
G(z, w) = (-z^2 + \psi(w), \Phi(z, w)),
\]
where \(\psi \in \mathbb{C}[[\zeta]]\) and \(\Phi \in \mathbb{C}[z, w]\) are arbitrary power series of order at least 3.

If instead \(\rho \neq 1\) (recalling that \(\rho \neq 0\) too) then
\[
\text{Im } L|_{\mathcal{H}^d} = \text{Span} \left( 2\rho u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1} \right),
\]
and hence
\[
(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span} \left( u_{d+1,0}, (\rho - 1)(d + 1)u_{d+1,1} - 2\rho v_{d+1,0} \right).
\]

It then follows that every formal germ of the form
\[
F(z, w) = (-\rho z^2 + O_3, (1 - \rho)zw + O_3)
\]
with \(\rho \neq 0, 1\) is formally conjugated to a germ of the form
\[
G(z, w) = (-\rho z^2 + (\rho - 1)zw\varphi'(w) + \psi(w), (1 - \rho)zw - 2\rho \varphi(z)),
\]
where \(\varphi, \psi \in \mathbb{C}[[\zeta]]\) are arbitrary power series of order at least 3.

• Case \((2_{11\rho})\).
In this case we have
\[
\begin{align*}
L(u_{d,j}) &= -2\rho u_{d+1,j+1} - u_{d+1,j} - (1 + \rho)v_{d+1,j} \\
L(v_{d,j}) &= -u_{d+1,j+1} - 2v_{d+1,j} - (1 + \rho)v_{d+1,j+1}
\end{align*}
\]
\[(3.1)\]
for all \(d \geq 2\) and \(j = 0, \ldots, d\). We clearly have two subcases to consider.

If \(\rho = -1\) then
\[
\text{Im } L|_{\mathcal{H}^d} = \text{Span} \left( u_{d+1,0} - 2u_{d+1,1}, \ldots, u_{d+1,d} - 2u_{d+1,d+1}, u_{d+1,1} + 2v_{d+1,0}, \ldots, u_{d+1,d} + 2v_{d+1,d} \right),
\]
and hence
\[
(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span} \left( \sum_{j=0}^{d+1} \binom{d+1}{j} \frac{1}{2^j} \left( u_{d+1,j} - \frac{1}{4} v_{d+1,j+1} \right), v_{d+1,d+1} \right)
\]
\[
= \text{Span} \left( \left( \frac{1}{2} - w \right)^{d+1}, -\frac{1}{4} \left( \frac{1}{2} + w \right)^{d+1}, (0, z^{d+1}) \right).
\]
It then follows that every formal germ of the form
\[ F(z, w) = (-z^2 + zw + O_3, w^2 + O_3) \]
is formally conjugated to a germ of the form
\[ G(z, w) = \left( -z^2 + zw + \varphi \left( \frac{z}{2} + w \right), w^2 - \frac{1}{4} \varphi \left( \frac{z}{2} + w \right) + \psi(z) \right), \]
where \( \varphi, \psi \in \mathbb{C}[\zeta] \) are arbitrary power series of order at least 3.

If instead \( \rho \neq -1 \) (recalling that \( \rho \neq 0 \) too) then a basis of \( \text{Im} \, L |_{\mathcal{H}^d} \) is given by the vectors listed in (3.1), and a computation shows that \( \text{Im} \, L |_{\mathcal{H}^d}^\perp \) is given by homogeneous maps of the form
\[
\sum_{j=0}^{d+1} (a_j u_{d+1,j} + b_j v_{d+1,j})
\]
where the coefficients \( a_j, b_j \) satisfy the following relations:
\[
\begin{align*}
c_j b_j &= -\frac{2}{1+\rho} c_{j-1} b_{j-1} - \frac{1}{\rho(1+\rho)} c_{j-2} b_{j-2} \quad \text{for } j = 2, \ldots, d+1, \\
c_j a_j &= \frac{1}{\rho} c_{j-2} b_{j-2} \quad \text{for } j = 2, \ldots, d+1, \\
a_0 &= (3\rho - 1)b_0 + 2\rho(1+\rho) b_1, \\
a_1 &= -2(d+1)b_0 - (1+\rho)b_1,
\end{align*}
\]
where \( c_j^{-1} = \binom{d+1}{j} \) and \( b_0, b_1 \in \mathbb{C} \) are arbitrary. Solving this recurrence equation one gets
\[
b_j = \frac{1}{2\sqrt{-\rho}} \binom{d+1}{j} \left[ \frac{\rho(1+\rho)}{d+1} (m_{\rho}^j - n_{\rho}^j) b_1 + \left( \rho (m_{\rho}^j - n_{\rho}^j) + \sqrt{-\rho} (m_{\rho}^j + n_{\rho}^j) \right) b_0 \right],
\]
where \( \sqrt{-\rho} \) is any square root of \( -\rho \), and
\[
m_{\rho} = \frac{\sqrt{-\rho} - \rho}{\rho(1+\rho)}, \quad n_{\rho} = \frac{\sqrt{-\rho} + \rho}{\rho(1+\rho)}.
\]
It follows that the renormalized normal form of a formal germ of the form
\[ F(z, w) = (\rho z^2 + zw + O_3, (1+\rho)zw + w^2 + O_3) \]
with \( \rho \neq 0, -1 \) is
\[ G(z, w) = \left( \rho z^2 + zw + \frac{1}{\rho} \left[ 1 - \sqrt{-\rho} \phi(m_{\rho} z + w) + \frac{1 + \sqrt{-\rho}}{2n_{\rho}^2} \phi(n_{\rho} z + w) \right] + \frac{1 + \rho}{2\sqrt{-\rho}} \left( \frac{1}{m_{\rho}^2} \psi(m_{\rho} z + w) - \frac{1}{n_{\rho}^2} \psi(n_{\rho} z + w) \right), \right. \]
\[ (1+\rho)zw + w^2 + \frac{1 - \sqrt{-\rho}}{2} \phi(m_{\rho} z + w) + \frac{1 + \sqrt{-\rho}}{2} \phi(n_{\rho} z + w) + \rho(1+\rho) \left( \psi(m_{\rho} z + w) - \psi(n_{\rho} z + w) \right) \]
where \( \varphi, \psi \in \mathbb{C}[\zeta] \) are power series of order at least 3.
It then follows that every formal germ of the form
\[ L(u_{d,j}) = u_{d+1,j} - 2u_{d+1,j+1} \quad \text{and} \quad L(v_{d,j}) = u_{d+1,j+1} \]
for all \( d \geq 2 \) and \( j = 0, \ldots, d \). It follows that
\[ \text{Im} \ L|_{\mathcal{H}^d} = \text{Span} (u_{d+1,0}, \ldots, u_{d+1,d+1}) \]
and hence
\[ (\text{Im} \ L|_{\mathcal{H}^d})^\perp = \text{Span} (v_{d+1,0}, \ldots, v_{d+1,d+1}) \, . \]

It then follows that every formal germ of the form
\[ F(z, w) = (z^2 - zw + O_3, O_3) \]
is formally conjugated to a germ of the form
\[ G(z, w) = (z^2 - zw, \Phi(z, w)) \, , \]
where \( \Phi \in \mathbb{C}[z, w] \) is a power series of order at least 3.

- **Case (3_{100}).**

  In this case we have
  \[
  \begin{cases}
  L(u_{d,j}) = \rho(2u_{d+1,j+1} - u_{d+1,j}) + (\rho - 1)v_{d+1,j} \\
  L(v_{d,j}) = -\rho u_{d+1,j+1} + (\rho - 1)(v_{d+1,j+1} - 2v_{d+1,j})
  \end{cases}
  \]  \quad (3.2)

  for all \( d \geq 2 \) and \( j = 0, \ldots, d \). Then a basis of \( \text{Im} \ L|_{\mathcal{H}^d} \) is given by the homogeneous maps listed in (3.2), and a computation shows that \( (\text{Im} \ L|_{\mathcal{H}^d})^\perp \) is given by homogeneous maps of the form
  \[
  \sum_{j=0}^{d+1} (a_j u_{d+1,j} + b_j v_{d+1,j})
  \]
  where the coefficients \( a_j, b_j \) satisfy the following relations:
  \[
  \begin{align*}
  c_{j+1}a_{j+1} &= \frac{\rho - 1}{\rho} (c_{j+1}b_{j+1} - 2c_j b_j) \quad \text{for} \ j = 0, \ldots, d, \\
  c_{j+1}b_{j+1} &= 2c_j b_j - c_{j-1} b_{j-1} \quad \text{for} \ j = 1, \ldots, d, \\
  c_0 a_0 &= 2c_1 a_1 + \frac{\rho - 1}{\rho} c_0 b_0 .
  \end{align*}
  \]
  where \( c_j^{-1} = \binom{d+1}{j} \) and \( b_0, b_1 \in \mathbb{C} \) are arbitrary. Solving this recurrence equation we find
  \[
  \begin{align*}
  b_j &= \binom{d+1}{j} \left[ \frac{j}{d+1} b_1 - (j - 1)b_0 \right] \quad \text{for} \ j = 0, \ldots, d + 1, \\
  a_j &= \frac{\rho - 1}{\rho} \binom{d+1}{j} \left[ \frac{2-j}{d+1} b_1 + (j - 3)b_0 \right] \quad \text{for} \ j = 0, \ldots, d + 1,
  \end{align*}
  \]
  where \( b_0, b_1 \in \mathbb{C} \) are arbitrary. So every formal germ of the form
  \[ F(z, w) = (\rho(-z^2 + zw) + O_3, (1 - \rho)(zw - w^2) + O_3) \]
  with \( \rho \neq 0, 1 \) is formally conjugated to a germ of the form
  \[ G(z, w) = \left( \rho(-z^2 + zw) + z \frac{\partial}{\partial z} [\varphi(z + w) + \psi(z + w)] - \varphi(z + w), \right.\]
  \[ \left. (1 - \rho)(zw - w^2) + \frac{\rho - 1}{\rho} \left( z \frac{\partial}{\partial z} [\varphi(z + w) - \psi(z + w)] - 3\varphi(z + w) + 2\psi(z + w) \right) \right) \]
  where \( \varphi, \psi \in \mathbb{C}[\zeta] \) are power series of order at least 3.
• **Case** \((3\rho \tau_1)\).

In this case we have

\[
L(u_{d,j}) = (\tau - 1)u_{d+1,j} + 2\rho u_{d+1,j+1} + (\rho - 1)v_{d+1,j}
\]

and

\[
L(v_{d,j}) = (\tau - 1)v_{d+1,j+1} + 2\tau v_{d+1,j} + (\rho - 1)v_{d+1,j+1}
\]

for all \(d \geq 2\) and \(j = 0, \ldots, d\). As before, we have a few subcases to consider.

Assume first \(\rho = \tau = 1\). Then

\[
\text{Im} \left. L \right|_{H^d} = \text{Span} \left( u_{d+1,1}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d} \right);
\]

hence

\[
\left( \text{Im} \left. L \right|_{H^d} \right)^\perp = \text{Span} \left( u_{d+1,0}, v_{d+1,d+1} \right),
\]

It then follows that every formal germ of the form

\[
F(z, w) = (-z^2 + O_3, -w^2 + O_3)
\]

is formally conjugated to a germ of the form

\[
G(z, w) = (-z^2 + \varphi(w), -w^2 + \psi(z)),
\]

where \(\varphi, \psi \in \mathbb{C}[z, w]\) are arbitrary power series of order at least 3.

Assume now \(\rho \neq 1\). Then a computation shows that \(\left( \text{Im} \left. L \right|_{H^d} \right)^\perp\) is given by homogeneous maps of the form

\[
\sum_{j=0}^{d+1} (a_j u_{d+1,j} + b_j v_{d+1,j})
\]

where the coefficients \(a_j, b_j\) satisfy the following relations:

\[
\begin{cases}
    c_{j+1}a_{j+1} = \frac{\tau}{\rho} c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\
    c_{j+1}b_{j+1} = -\frac{2\tau}{\rho - 1} c_j b_j - \frac{\tau(\tau - 1)}{\rho(\rho - 1)} c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d,
\end{cases}
\]

\[
(\tau - 1)c_1 a_1 + (\rho - 1)c_1 b_1 + 2\tau c_0 b_0 = 0,
\]

\[
(\tau - 1)c_0 a_0 + (\rho - 1)c_0 b_0 + 2\rho c_1 a_1 = 0,
\]

where \(c_j^{-1} = \binom{d+1}{j}\) and \(b_0, b_1 \in \mathbb{C}\) are arbitrary.

When \(\tau = 1\) conditions (3.3) reduce to

\[
\begin{cases}
    c_{j+1}a_{j+1} = \frac{1}{\rho} c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\
    c_{j+1}b_{j+1} = -\frac{2}{\rho - 1} c_j b_j & \text{for } j = 1, \ldots, d,
\end{cases}
\]

\[
(\rho - 1)c_1 b_1 + 2c_0 b_0 = 0,
\]

\[
(\rho - 1)c_0 b_0 + 2\rho c_1 a_1 = 0,
\]
whose solution is

\[
\begin{align*}
  a_j &= \binom{d+1}{j} \frac{1}{\rho} \left( \frac{2}{1-\rho} \right)^{j-2} \ b_0 \quad \text{for } j = 1, \ldots, d + 1, \\
  b_j &= \binom{d+1}{j} \left( \frac{2}{1-\rho} \right)^{j} \ b_0 \quad \text{for } j = 0, \ldots, d + 1,
\end{align*}
\]

where \(a_0, b_0 \in \mathbb{C}\) are arbitrary. Therefore

\[
(\Im L|_{y^d})^\perp = \text{Span} \left( (w^{d+1}, 0), \left( \frac{(1-\rho)^2}{4\rho} \left( \frac{2}{1-\rho} z + w \right)^{d+1}, \left( \frac{2}{1-\rho} z + w \right)^{d+1} \right) \right),
\]

and thus every formal germ of the form

\[
F(z, w) = (-\rho z^2 + O_3, (1-\rho)zw - w^2 + O_3)
\]

with \(\rho \neq 1\) is formally conjugated to a germ of the form

\[
G(z, w) = \left( -\rho z^2 + \varphi(w) + \frac{(1-\rho)^2}{4\rho} \psi \left( \frac{2}{1-\rho} z + w \right), (1-\rho)zw - w^2 + \psi \left( \frac{2}{1-\rho} z + w \right) \right),
\]

where \(\varphi, \psi \in \mathbb{C}[\zeta]\) are arbitrary power series of order at least 3.

The case \(\rho = 1\) and \(\tau \neq 1\) is treated in the same way; we get that every formal germ of the form

\[
F(z, w) = (-z^2 + (1-\tau)zw + O_3, -\tau w^2 + O_3)
\]

with \(\tau \neq 1\) is formally conjugated to a germ of the form

\[
G(z, w) = \left( -z^2 + (1-\tau)zw + \psi \left( \frac{1-\tau}{2} z + w \right), -\tau w^2 + \varphi(z) + \frac{(1-\tau)^2}{4\tau} \psi \left( \frac{1-\tau}{2} z + w \right) \right),
\]

where \(\varphi, \psi \in \mathbb{C}[\zeta]\) are arbitrary power series of order at least 3.

Finally assume \(\rho, \tau \neq 1\). Solving the recurrence equation (3.3) we find

\[
b_j = \frac{1}{2 \sqrt{\rho \tau (\rho + \tau - 1)}} \binom{d+1}{j} \left[ \frac{\rho(\rho - 1)}{d+1} (m_{\rho,\tau}^j - n_{\rho,\tau}^j) b_1 \right.
\]

\[+ \left. \left( \rho \tau (m_{\rho,\tau}^j - n_{\rho,\tau}^j) + \sqrt{\rho \tau (\rho + \tau - 1)} (m_{\rho,\tau}^j + n_{\rho,\tau}^j) \right) b_0 \right],
\]

for \(j = 0, \ldots, d + 1\), where \(\sqrt{\rho \tau (\rho + \tau - 1)}\) is any square root of \(\rho \tau (\rho + \tau - 1)\), and

\[
m_{\rho,\tau} = \frac{\sqrt{\rho \tau (\rho + \tau - 1) - \rho \tau}}{\rho(\rho - 1)}, \quad n_{\rho,\tau} = -\frac{\sqrt{\rho \tau (\rho + \tau - 1) + \rho \tau}}{\rho(\rho - 1)}.
\]

Moreover, from (3.3) we also get

\[
a_j = \frac{\tau}{2\rho \sqrt{\rho \tau (\rho + \tau - 1)}} \binom{d+1}{j} \left[ \frac{\rho(\rho - 1)}{d+1} (m_{\rho,\tau}^{j-2} - n_{\rho,\tau}^{j-2}) b_1 \right.
\]

\[+ \left. \left( \rho \tau (m_{\rho,\tau}^{j-2} - n_{\rho,\tau}^{j-2}) + \sqrt{\rho \tau (\rho + \tau - 1)} (m_{\rho,\tau}^{j-2} + n_{\rho,\tau}^{j-2}) \right) b_0 \right],
\]
again for \(j = 0, \ldots, d + 1\). It follows that the renormalized normal form of a formal germ of the form
\[
F(z, w) = (-\rho z^2 + (1 - \tau)zw + O_3, (1 - \rho)zw - \tau w^2 + O_3)
\]
with \(\rho, \tau \neq 0, 1\) and \(\rho + \tau \neq 1\), is
\[
G(z, w) = \left(-\rho z^2 + (1 - \tau)zw + \frac{\tau}{\rho} \left[ \frac{\sqrt{\rho + \tau - 1} + \sqrt{\rho \tau}}{2m_{\rho, \tau}^2} \varphi(m_{\rho, \tau}z + w) 
+ \frac{\sqrt{\rho + \tau - 1} - \sqrt{\rho \tau}}{2n_{\rho, \tau}^2} \varphi(n_{\rho, \tau}z + w) 
+ \frac{1}{m_{\rho, \tau}^2} \psi(m_{\rho, \tau}z + w) - \frac{1}{n_{\rho, \tau}^2} \psi(n_{\rho, \tau}z + w) \right] 
+ \frac{\sqrt{\rho + \tau - 1} + \sqrt{\rho \tau}}{2} \varphi(m_{\rho, \tau}z + w) 
+ \frac{\sqrt{\rho + \tau - 1} - \sqrt{\rho \tau}}{2} \varphi(n_{\rho, \tau}z + w) 
+ \psi(m_{\rho, \tau}z + w) - \psi(n_{\rho, \tau}z + w) \right)
\]
\((1 - \rho)zw - \tau w^2 + \frac{\sqrt{\rho + \tau - 1} + \sqrt{\rho \tau}}{2} \varphi(m_{\rho, \tau}z + w) 
+ \frac{\sqrt{\rho + \tau - 1} - \sqrt{\rho \tau}}{2} \varphi(n_{\rho, \tau}z + w) 
+ \psi(m_{\rho, \tau}z + w) - \psi(n_{\rho, \tau}z + w) \right),
\]
where the square roots of \(\rho \tau\) and of \(\rho + \tau - 1\) are chosen so that their product is equal to the previously chosen square root of \(\rho \tau (\rho + \tau - 1)\), and \(\varphi, \psi \in \mathbb{C}[\xi]\) are power series of order at least 3.

4. Examples with \(\Lambda = I\)
In this section we shall assume \(n = \mu = 2\) and \(\Lambda = I\), that is we shall be interested in 2-dimensional germs tangent to the identity of order 2. We shall keep using the notations introduced in the previous section. It should be recall that in his monumental work [É1] (see [É2] for a survey) Écalle studied the formal classification of germs tangent to the identity in dimension \(n\) giving a complete set of formal invariants for germs satisfying a generic condition: the existence of at least one non-degenerate characteristic direction (an eigenradius, in Écalle’s terminology). A characteristic direction of a germ tangent to the identity \(F\) is a non-zero direction \(v\) such that \(F_\mu(v) = \lambda v\) for some \(\lambda \in \mathbb{C}\), where \(F_\mu\) is the first (nonlinear) non-vanishing term in the homogeneous expansion of \(F\). The characteristic direction \(v\) is degenerate if \(\lambda = 0\).

For this reason, we decided to discuss here the cases without non-degenerate characteristic directions, that is the cases \((1_{00}), (1_{10})\) and \((2_{001})\), that cannot be dealt with Écalle’s methods. Furthermore, we shall also study the somewhat special case \((\infty)\), where all directions are characteristic; and we shall examine in detail case \((2_{10p})\), where interesting second-order resonance phenomena appear.

When \(\Lambda = I\) the operator \(L = L_{F_2, \Lambda}\) is given by
\[
L(H) = \text{Jac}(H) \cdot F_2 - \text{Jac}(F_2) \cdot H.
\]

• Case \((\infty)\).
In this case we have
\[
L(u_{d,j}) = (d - 2)u_{d+1,j+1} - v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = (d - 1)v_{d+1,j+1}
\]
for all \(d \geq 2\) and \(j = 0, \ldots, d\). Therefore
\[
\text{Im} \ L|_{\mathcal{H}^d} = \begin{cases} \text{Span} \ (u_{d+1,1}, \ldots, u_{d+1,d+1}, (d - 2)u_{d+1,1} - v_{d+1,0}, v_{d+1,1}, \ldots, v_{d+1,d+1}) & \text{for } d > 2, \\ \text{Span} \ (v_3,0, \ldots, v_3,3) & \text{for } d = 2. \end{cases}
\]
Thus
\[(\text{Im } L|_{\mathcal{H}^d})^\perp = \begin{cases} \text{Span } (u_{d+1,0}, (d+1)u_{d+1,1} + (d-2)v_{d+1,0}) & \text{for } d > 2, \\
\text{Span } (u_{3,0}, \ldots, u_{3,3}) & \text{for } d = 2. \end{cases}\]
It then follows that every formal power series of the form
\[F(z, w) = (z + z^2 + O_3, w + zw + O_3)\]
is formally conjugated to a power series of the form
\[G(z, w) = (z + z^2 + a_0 z^3 + a_1 z^2 w + a_2 zw^2 + \varphi(w) + zw \psi(w), w + zw \psi(w) - 3\psi(w))\]
where \(\varphi \in \mathbb{C}[\zeta]\) is an arbitrary power series of order at least 3, \(\psi \in \mathbb{C}[\zeta]\) is an arbitrary power series of order at least 4 and \(a_0, a_1, a_2 \in \mathbb{C}\).

\textit{Case (1_0).}
In this case we have
\[L(u_{d,j}) = (j - d)u_{d+1,j+2} + 2v_{d+1,j+1} \quad \text{and} \quad L(v_{d,j}) = (j - d)v_{d+1,j+2}\]
for all \(d \geq 2\) and \(j = 0, \ldots, d\). Therefore
\[(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span } (2v_{d+1,1}, u_{d+1,2}, u_{d+1,3}, \ldots, u_{d+1,d+1}, v_{d+1,2}, \ldots, v_{d+1,d+1})\]
and thus
\[(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span } (u_{d+1,0}, u_{d+1,1}, v_{d+1,0}, u_{d+1,2} + v_{d+1,1})\]
It then follows that every formal power series of the form
\[F(z, w) = (z + O_3, w - z^2 + O_3)\]
is formally conjugated to a power series of the form
\[G(z, w) = (z + w \varphi_2(w) + z^2 \psi(w), w - z^2 + w \varphi_3(w) + zw \psi(w))\]
where \(\varphi_1, \varphi_2, \varphi_3 \in \mathbb{C}[\zeta]\) are arbitrary power series of order at least 2, and \(\psi \in \mathbb{C}[\zeta]\) is an arbitrary power series of order at least 1.

\textit{Case (1_1).}
In this case we have
\[L(u_{d,j}) = (2 - d)u_{d+1,j+1} - (d - j)u_{d+1,j+2} + 2v_{d+1,j+1} + v_{d+1,j}\]
and
\[L(v_{d,j}) = (1 - d)v_{d+1,j+1} - (d - j)v_{d+1,j+2}\]
for all \(d \geq 2\) and \(j = 0, \ldots, d\). Therefore
\[(\text{Im } L|_{\mathcal{H}^d})^\perp = \begin{cases} \text{Span } ((2 - d)u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}) & \text{for } d > 2, \\
\text{Span } (v_{3,0} - 2u_{3,2}, u_{3,3}, v_{3,1}, v_{3,2}, v_{3,3}) & \text{for } d = 2. \end{cases}\]
and thus
\[(\text{Im } L|_{\mathcal{H}^d})^\perp = \begin{cases} \text{Span } (u_{d+1,0}, (d + 1)u_{d+1,1} + (d - 2)v_{d+1,0}) & \text{for } d > 2, \\
\text{Span } (u_{3,0}, u_{3,1}, 3u_{3,2} + 2v_{3,0}) & \text{for } d = 2. \end{cases}\]
It then follows that every formal power series of the form
\[F(z, w) = (z - z^2 + O_3, w - zw + O_3)\]
is formally conjugated to a power series of the form
\[G(z, w) = (z - z^2 + \varphi(w) + a_1 zw^2 + 3a_2 z^2 w + z\psi'(w), w - z^2 - zw + 2a_2 w^3 + w\psi'(w) - 3\psi(w))\]
where \(\varphi \in \mathbb{C}[\zeta]\) is an arbitrary power series of order at least 3, \(\psi \in \mathbb{C}[\zeta]\) is an arbitrary power series of order at least 4, and \(a_1, a_2 \in \mathbb{C}\).
In this case we have

\[ L(u_{d,j}) = (d - j)u_{d+1,j+1} - v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = (d - j - 1)v_{d+1,j+1} \]

for all \( d \geq 2 \) and \( j = 0, \ldots, d \). It follows that

\[ \text{Im} \, L|_{H^d} = \text{Span} \left\{ du_{d+1,1} - v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d}, v_{d+1,1}, \ldots, v_{d+1,d+1} \right\} \]

and hence

\[ (\text{Im} \, L|_{H^d})^+ = \text{Span} \left\{ u_{d+1,0}, u_{d+1,d+1}, (d + 1)u_{d+1,1} + dv_{d+1,0} \right\} \]

It then follows that every formal germ of the form

\[ F(z, w) = (z + O_3, w + zw + O_3) \]

is formally conjugated to a germ of the form

\[ G(z, w) = (z + \varphi_1(z) + \varphi_2(w) + z\psi'(w), zw + w\psi'(w) - \psi(w)) \]

where \( \varphi_1, \varphi_2, \psi \in \mathbb{C}[\zeta] \) are arbitrary power series of order at least 3.

- **Case (2_001).**
  
  In this case we have

  \[ L(u_{d,j}) = (d - j)u_{d+1,j+1} - v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = (d - j - 1)v_{d+1,j+1} \]

  for all \( d \geq 2 \) and \( j = 0, \ldots, d \). Here we can see the resonance phenomena we mentioned at the beginning of this section: for some values of \( \rho \) the dimension of the kernel of \( L|_{H^d} \) can increase, and in some cases we shall end up with a normal form depending on power series evaluated in monomials of the form \( z^{a-b} w^a \).

  Let us put

  \[ E_d = \left\{ \frac{d - j - 1}{d - 1} \left\vert j = 0, \ldots, d \right\} \setminus \{0\} \quad \text{and} \quad F_d = \left\{ \frac{d - j}{d - 2} \left\vert j = 0, \ldots, d - 1 \right\} \right\} \]

  (we are excluding 0 because \( \rho \neq 0 \) by assumption), where \( E_d \) is defined for all \( d \geq 2 \) whereas \( F_d \) is defined for all \( d \geq 3 \), and set

  \[ \mathcal{E} = \bigcup_{d \geq 2} E_d = ((0, 1] \cap \mathbb{Q}) \cup \left\{ \frac{1}{n} \left\vert n \in \mathbb{N}^* \right\} \right\} \]

  and

  \[ \mathcal{F} = \bigcup_{d \geq 3} F_d = ((0, 1] \cap \mathbb{Q}) \cup \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} \left\vert n \in \mathbb{N}^* \right\} \right\} \]

  So \( \mathcal{E} \) is the set of \( \rho \in \mathbb{C}^* \) such that \( L(v_{d,j}) = 0 \) for some \( d \geq 2 \) and \( 0 \leq j \leq d \), while \( \mathcal{F} \) is the set of \( \rho \in \mathbb{C}^* \) such that \( L(u_{d,j}) = (\rho - 1)v_{d+1,j} \) for some \( d \geq 3 \) and \( 0 \leq j \leq d - 1 \).

  Let us first discuss the non-resonant case, when \( \rho \not\in \mathcal{E} \cup \mathcal{F} \). Then none of the coefficients in (4.1) vanish, and thus

  \[ \text{Im} \, L|_{H^d} = \text{Span} \left\{ (d - d\rho + 2\rho)u_{d+1,1}, (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1} \right\} \]
Taking care of the case

and hence

\[
(\text{Im } L|_{\mathcal{H}^d})^\perp = \text{Span } (u_{d+1,0}, (1 - \rho)(d + 1)u_{d+1,1} + (d(1 - \rho) + 2\rho)v_{d+1,0})
\]

It then follows that every formal germ of the form

\[
F(z, w) = (z - \rho z^2 + O_3, w + (1 - \rho)z w + O_3)
\]

with \(\rho \notin \mathcal{E} \cup \mathcal{F}\) (and \(\rho \neq 0\)) is formally conjugated to a germ of the form

\[
G(z, w) = (z - \rho z^2 + \varphi(w) + (1 - \rho)z\psi(w), w + (1 - \rho)zw + (1 - \rho)w\psi(w) + (3\rho - 1)\psi(z))
\]

where \(\varphi, \psi \in \mathbb{C}[\zeta]\) are arbitrary power series of order at least 3.

Assume now \(\rho \in \mathcal{F} \setminus \mathcal{E}\). Then \(L(v_{d,j}) \neq O\) always, and thus \(v_{d+1,j} \in \text{Im } L|_{\mathcal{H}^d}\) for all \(d \geq 2\) and all \(j = 1, \ldots, d + 1\). Since \(\rho > 1\), if \(d > 2\) it also follows that \(u_{d+1,j+1} \in \text{Im } L|_{\mathcal{H}^d}\) for \(j = 1, \ldots, d\).

Now, if \(\rho = 1 + (1/n)\) then

\[
\frac{d}{d - 2} = \rho \iff d = 2(n + 1),
\]

and

\[
\frac{d - 1}{d - 2} = \rho \iff d = n + 2.
\]

Taking care of the case \(d = 2\) separately, we then have

\[
\text{Im } L|_{\mathcal{H}^d} = \begin{cases}
\text{Span } ((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}) & \text{for } d \geq 3, \ d \neq n + 2, \ 2(n + 1), \\
\text{Span } (u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,3}, \ldots, u_{d+1,d+1}, v_{d+1,1} + 1, \ldots, v_{d+1,d+1}) & \text{for } d = n + 2, \\
\text{Span } (u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,0}, v_{d+1,1}, \ldots, v_{d+1,d+1}) & \text{for } d = 2(n + 1), \\
\text{Span } (2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}) & \text{for } d = 2,
\end{cases}
\]

and hence

\[
(\text{Im } L|_{\mathcal{H}^d})^\perp = \begin{cases}
\text{Span } (u_{d+1,0}, (1 - \rho)(d + 1)u_{d+1,1} + (d(1 - \rho) + 2\rho)v_{d+1,0}) & \text{for } d \geq 3, \ d \neq n + 2, \ 2(n + 1), \\
\text{Span } (u_{d+1,0}, u_{d+1,2}, (1 - \rho)(d + 1)u_{d+1,1} + v_{d+1,0}) & \text{for } d = n + 2, \\
\text{Span } (u_{d+1,0}, u_{d+1,1}) & \text{for } d = 2(n + 1), \\
\text{Span } (u_{3,0}, u_{3,3}, 3(1 - \rho)u_{3,1} + 2v_{3,0}) & \text{for } d = 2.
\end{cases}
\]

It then follows that every formal germ of the form

\[
F(z, w) = \left( z - \left( 1 + \frac{1}{n} \right) z^2 + O_3, w - \frac{1}{n} zw + O_3 \right)
\]

with \(n \in \mathbb{N}^*\) is formally conjugated to a germ of the form

\[
G(z, w) = \left( z - \left( 1 + \frac{1}{n} \right) z^2 + \varphi(w) + (1 - \rho)z\psi(w) + a_0 z^3 + a_1 z^2 w^{n+1}, \\
w - \frac{1}{n} zw + (1 - \rho)w\psi(w) + (3\rho - 1)\psi(w) \right),
\]
where $\varphi, \psi \in \mathbb{C}[\zeta]$ are arbitrary power series of order at least 3, and $a_0, a_1 \in \mathbb{C}$.

If instead $\rho = 1 + (2/m)$ with $m$ odd (if $m$ is even we are again in the previous case) then

$$\frac{d}{d - 2} = \rho \iff d = m + 2,$$

whereas $\frac{d-1}{d-2} \neq \rho$ always. Hence

$$\text{Im } L|_{\mathcal{T}^d} = \left\{ \begin{array}{ll}
\text{Span } ((d - dp + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}) & \text{for } d \geq 3, d \neq m+2, \\
\text{Span } (u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d+1}) & \text{for } d = m+2, \\
\text{Span } (2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}) & \text{for } d = 2,
\end{array} \right.$$}

and thus

$$\text{Im } L|_{\mathcal{T}^d}^\perp = \left\{ \begin{array}{ll}
\text{Span } (u_{d+1,0}, (1 - \rho)(d+1)u_{d+1,1} + (d - dp + 2\rho)v_{d+1,0}) & \text{for } d \geq 3, d \neq m+2, \\
\text{Span } (u_{d+1,0}, u_{d+1,1}) & \text{for } d = m+2, \\
\text{Span } (u_{3,0}, u_{3,3}, 3(1 - \rho)u_{3,1} + 2v_{3,0}) & \text{for } d = 2.
\end{array} \right.$$}

It then follows that every formal germ of the form

$$F(z, w) = \left( z - \left(1 + \frac{2}{m}\right) z^2 + O_3, w - \frac{2}{m} zw + O_3 \right)$$

with $m \in \mathbb{N}^*$ odd is formally conjugated to a germ of the form

$$G(z, w) = \left( z - \left(1 + \frac{2}{m}\right) z^2 + \varphi(w) + a_0 z^3 + (1 - \rho)z(w \psi'(w) + \psi(w)), \right.$$  
$$w - \frac{2}{m} zw + (1 - \rho)w^2 \psi'(w) + 2\rho w \psi(w) \),$$

where $\varphi \in \mathbb{C}[\zeta]$ is an arbitrary power series of order at least 3, $\psi \in \mathbb{C}[\zeta]$ is an arbitrary power series of order at least 2, and $a_0, a_1 \in \mathbb{C}$.

Now let us consider the case $\rho = -1/n \in \mathcal{E} \setminus \mathcal{F}$. In this case the coefficients in the expression of $L(u_{d,j})$ are always different from zero (with the exception of $d = j = 2$), whereas

$$d - j - dp + \rho - 1 = 0 \iff j = d = n + 1.$$  

It follows that

$$\text{Im } L|_{\mathcal{T}^d} = \left\{ \begin{array}{ll}
\text{Span } ((d - dp + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}) & \text{for } d \geq 3, d \neq n+1, \\
\text{Span } ((d - dp + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d}) & \text{for } d = n+1, \\
\text{Span } (2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}) & \text{for } d = 2,
\end{array} \right.$$}

and thus

$$\text{Im } L|_{\mathcal{T}^d}^\perp = \left\{ \begin{array}{ll}
\text{Span } (u_{d+1,0}, (1 - \rho)(d+1)u_{d+1,1} + (d - dp + 2\rho)v_{d+1,0}) & \text{for } d \geq 3, d \neq n+1, \\
\text{Span } (u_{d+1,0}, v_{d+1,1}, (1 - \rho)(d+1)u_{d+1,1} + (d - dp + 2\rho)v_{d+1,0}) & \text{for } d = n+1, \\
\text{Span } (u_{3,0}, u_{3,3}, 3(1 - \rho)u_{3,1} + 2v_{3,0}) & \text{for } d = 2.
\end{array} \right.$$
It then follows that every formal germ of the form
\[
F(z, w) = \left( z + \frac{1}{n} z^2 + O_3, w + \left( 1 + \frac{1}{n} \right) zw + O_3 \right)
\]
with \( n \in \mathbb{N}^* \) is formally conjugated to a germ of the form
\[
G(z, w) = \left( z + \frac{1}{n} z^2 + \varphi(w) + a_0 z^3 + (1 - \rho) z(w\psi'(w) + \psi(w)), \\
\right.
\]
\[
\left. w + \left( 1 + \frac{1}{n} \right) zw + \psi(z) + a_1 z^{n+2} + (1 - \rho) w^2 \psi'(w) + 2\rho w\psi(w) \right),
\]
where \( \varphi \in \mathbb{C}[\zeta] \) is an arbitrary power series of order at least 3, \( \psi \in \mathbb{C}[\zeta] \) is an arbitrary power series of order at least 2, and \( a_0, a_1 \in \mathbb{C} \).

Let us now discuss the extreme case \( \rho = 1 \). It is clear that
\[
\text{Im } L \big|_{\mathcal{H}^d} = \text{Span} \left( u_{d+1,1}, u_{d+1,2}, u_{d+1,4}, \ldots, u_{d+1,d+1}, v_{d+1,2}, \ldots, v_{d+1,d+1} \right),
\]
and hence
\[
\left( \text{Im } L \big|_{\mathcal{H}^d} \right)^\perp = \text{Span} \left( u_{d+1,0}, u_{d+1,3}, v_{d+1,0}, v_{d+1,1} \right),
\]
It then follows that every formal germ of the form
\[
F(z, w) = \left( z - z^2 + O_3, w + O_3 \right)
\]
is formally conjugated to a germ of the form
\[
G(z, w) = \left( z - z^2 + \varphi_1(w) + z^3 \psi(w), w + \varphi_2(w) + z\varphi_3(w) \right),
\]
where \( \varphi_1, \varphi_2 \in \mathbb{C}[\zeta] \) are arbitrary power series of order at least 3, \( \varphi_3 \in \mathbb{C}[\zeta] \) is an arbitrary power series of order at least 2, and \( \psi \in \mathbb{C}[\zeta] \) is an arbitrary power series.

We are left with the case \( \rho \in (0,1) \cap \mathbb{Q} \). Write \( \rho = a/b \) with \( a, b \in \mathbb{N} \) coprime and \( 0 < a < b \). Now
\[
d - j - 1 - \frac{a}{b}(d - 1) = 0 \iff j = \frac{(d - 1)(b - a)}{b};
\]
since \( a \) and \( b \) are coprime, this happens if and only if \( d = b\ell + 1 \) and \( j = (b - a)\ell \) for some \( \ell \geq 1 \).

Analogously,
\[
d - j - \frac{a}{b}(d - 2) = 0 \iff j = d - \frac{a(d - 2)}{b};
\]
again, being \( a \) and \( b \) coprime, this happens if and only if \( d = b\ell + 2 \) and \( j = (b - a)\ell + 2 \) for some \( \ell \geq 0 \). It follows that
\[
\text{Im } L \big|_{\mathcal{H}^d} = \begin{cases} 
\text{Span} \left( (d - dp + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1} \right) & \text{for } d \geq 3, d \not\equiv 1, 2 \text{ (mod } b) \\
\text{Span} \left( (d - dp + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}, \frac{\rho}{b}u_{d+1,0}(b-a)\ell+2, \ldots, u_{d+1,d+1}, \frac{\rho}{b}u_{d+1,0}(b-a)\ell+2 - \left( \frac{\rho}{b} - 1 \right) u_{d+1,0}(b-a)\ell+1 \right) & \text{for } d = b\ell + 1, \\
\text{Span} \left( (d - dp + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,(b-a)\ell+3}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1} \right) & \text{for } d = b\ell + 2, \\
\text{Span} \left( 2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3} \right) & \text{for } d = 2,
\end{cases}
\]
Formal Poincaré-Dulac renormalization for holomorphic germs

(Im $L|_{3^d}$)

$$\left\{\begin{array}{ll}
\text{Span} (u_{d+1,0}, (1 - \rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}) & \text{for } d \geq 3, d \neq 1, 2 \pmod{b}, \\
\text{Span} (u_{d+1,0}, (1 - \rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}, \\
(\ell - a)(\ell + 1)u_{d+1,(b-a)\ell+2} + a((b-a)\ell + 2)v_{d+1,(b-a)\ell+1}) & \text{for } d = b\ell + 1, \\
\text{Span} (u_{d+1,0}, u_{d+1,(b-a)\ell+3}, (1 - \rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}) & \text{for } d = b\ell + 2, \\
\text{Span} (u_{3,0}, u_{3,3}, 3(1 - \rho)u_{3,1} + 2v_{3,0}) & \text{for } d = 2.
\end{array}\right.$$  

It then follows that every formal germ of the form

$$F(z, w) = \left( z - \frac{a}{b} z^2 + O_3, w + \left( 1 - \frac{a}{b} \right) z w + O_3 \right)$$

with $a/b \in (0, 1) \cap \mathbb{Q}$ and $a$, $b$ coprime, is formally conjugated to a germ of the form

$$G(z, w) = \left( z - \frac{a}{b} z^2 + \varphi(w) + z^3 \varphi_0 (z^{b-a} w^a) + (b-a) \frac{\partial}{\partial w} (z^2 w \chi (z^{b-a} w^a)) + \left( 1 - \frac{a}{b} \right) z (w \psi'(w) + \psi(w)), \\
w + \left( 1 - \frac{a}{b} \right) z w + a \frac{\partial}{\partial z} (z^2 w \chi (z^{b-a} w^a)) + \left( 1 - \frac{a}{b} \right) w^2 \psi'(w) + 2 \frac{a}{b} w \psi(w) \right) ,$$

where $\varphi, \psi \in \mathbb{C}[\zeta]$ are arbitrary power series of order at least 3, and $\varphi_0, \chi \in \mathbb{C}[\zeta]$ are arbitrary power series of order at least 1.

References.


